Math 4513
Solutions to Homework 9

1. Use the Taylor series method to calculate a solution to

$$\frac{dx}{dt} = x^2 t \quad , \quad x(0) = 1$$

that's accurate to order t^4.

- We have

$$\begin{aligned}
x(0) &= 1 \\
x'(0) &= x^2 t\Big|_{t=0} = (1)^2(0) = 0 \\
x''(0) &= \left(2xx't + x^2\right)\Big|_{t=0} = 2(1)(0) + (1)^2 = 1 \\
x'''(0) &= \left(2(x')^2 t + 2xx''t + 4xx'\right)\Big|_{t=0} = 0 + 0 + 0 + 0 = 0 \\
x^{(iv)}(0) &= \left(4x'x''t + 2\left(x'\right)^2 + 2x'x''t + 2xx'''t + 2xx'' + 4(x')^2 + 4xx''\right)\Big|_{t=0} = 2 + 4 = 6
\end{aligned}$$

  so

$$\begin{aligned}
x(t) &= x(0) + x'(0)t + \frac{1}{2}x''(0)t^2 + \frac{1}{6}x'''(0)t^3 + \frac{1}{24}x^{(iv)}(0)t^4 + \mathcal{O}(t^5) \\
&= 1 + \frac{1}{2}t^2 + \frac{1}{4}t^4 + \mathcal{O}(t^5)
\end{aligned}$$

2. Use a five step Euler method to calculate the solution of

$$\frac{dx}{dt} = x^2 t \quad , \quad x(0) = 1$$

on the interval $[0, 1]$.

- To divide the interval $[0, 1]$ into five subintervals, we use a step size of

$$\Delta t = \frac{1-0}{5} = 0.2$$

$$
\begin{aligned}
t_0 &= 0 \\
x_0 &= 1 \\
m_0 &= x^2t\Big|_{t_0} = x_0t_0^2 = 0 \\
t_1 &= 0.2 \\
x_1 &= x_0 + m_0\Delta t = 1 + 0 = 1 \\
m_1 &= x_1^2t_1 = (1)^2(0.2) = 0.2 \\
t_2 &= 0.4 \\
x_2 &= x_1 + m_1\Delta t = 1.04 \\
m_2 &= x_2^2t_2 = .43264 \\
t_3 &= 0.6 \\
x_3 &= x_2 + m_2\Delta t = 1.126528000 \\
m_3 &= x_3^2t_3 = .7614392010 \\
t_4 &= 0.8 \\
x_4 &= x_3 + m_3\Delta t = 1.278815840 \\
m_4 &= x_4^2t_4 = 1.308295962 \\
t_5 &= 1.0 \\
x_5 &= x_4 + m_4\Delta t = 1.540475032
\end{aligned}
$$

This calculation could also be carried out using Maple:

```
n := 5;
t[0] := 0;
x[0] := 1;
f := (t,x) -> t*x^2;
dt := (1.0 - 0.0)/n;
for i from 0 to 4 do
   m[i] := f(t[i],x[i]);
   t[i+1] := t[i] + dt;
   x[i+1] := x[i] + m[i]*dt;
od;
```

3. Use a five step second order Runge-Kutta method to calculate the solution of

$$\frac{dx}{dt} = x^2t \quad , \quad x(0) = 1$$

on the interval $[0, 1]$.

- Again we have $t_0 = 0$, $x_0 = 1$, and $\Delta t = 0.2$. Successive values of ti and x_i are determined by computing

$$
\begin{aligned}
t_{i+1} &= t_i + \Delta t \\
F_{1,i} &= \Delta t f(t_i, x_i) \\
F_{2,i} &= \Delta t f\left(t_i + \Delta t, x_i + F_{1,i}\right) \\
x_{i+1} &= x_i + \frac{1}{2}(F_{1,i} + F_{2,i})
\end{aligned}
$$

The following Maple program automates this calculation.

```
#second order Runge-Kutta
```

```
n := 5;
t[0] := 0;
x[0] := 1;
f := (t,x) -> t*x^2;
dt := (1.0 - 0.0)/n;
for i from 0 to 4 do
   t[i+1] := t[i] + dt;
   F1[i] := dt*f(t[i],x[i]);
   F2[i] := dt*f(t[i]+dt,x[i]+F1[i]);
   x[i+1] := x[i] + (F1[i] + F2[i])/2;
od;
```

This program produces the following table of values

i	t	F1	F2	x
0	0.000000000	0.0000000000	0.0400000000	1.000000000
1	0.200000000	0.0416160000	0.0901622825	1.020000000
2	0.400000000	0.0943324182	0.1671507514	1.085889141
3	0.600000000	0.1776228388	0.3110308806	1.216630726
4	0.800000000	0.3415035304	0.6497732142	1.460957585
5	1.000000000			1.956595957

4. Use a five step fourth order Runge-Kutta method to calculate the solution of

$$\frac{dx}{dt} = x^2 t \quad , \quad x(0) = 1$$

on the interval $[0, 1]$.

- Again we have $t_0 = 0$, $x_0 = 1$, and $\Delta t = 0.2$. Successive values of ti and x_i are determined by computing

$$\begin{aligned} t_{i+1} &= t_i + \Delta t \\ F_{1,i} &= \Delta t f(t_i, x_i) \\ F_{2,i} &= \Delta t f\left(t_i + \Delta t, x_i + F_{1,i}\right) \\ x_{i+1} &= x_i + \frac{1}{2}(F_{1,i} + F_{2,i}) \end{aligned}$$

  The following Maple program automates this calculation.

```
#fourth order Runge-Kutta
n := 5;
t[0] := 0.0;
x[0] := 1.0
f := (t,x) -> t*x^2;
dt := (1.0 - 0.0)/n;
for i from 0 to 4 do
    t[i+1] := t[i] + dt;
    F1[i] := dt*f(t[i],x[i]);
    F2[i] := dt*f(t[i]+dt/2,x[i]+F1[i]/2);
    F3[i] := dt*f(t[i]+dt/2,x[i]+F2[i]/2);
    F4[i] := dt*f(t[i]+dt,x[i]+F3[i]);
    x[i+1] := x[i] + (F1[i] + 2*F2[i] + 2*F3[i] + F4[i])/6;
od;
```

This program calculates the following table

i	t	F1	F2	F3	F4	x
0	0.0000	0.000000	0.02000	0.02040	0.04165	1.0000
1	0.2000	0.041649	0.06505	0.06652	0.09451	1.0204
2	0.4000	0.094518	0.12865	0.13254	0.17846	1.0870
3	0.6000	0.17847	0.23980	0.25117	0.34607	1.2195
4	0.8000	0.34602	0.48626	0.52863	0.79938	1.4706
5	1.0000					1.9998

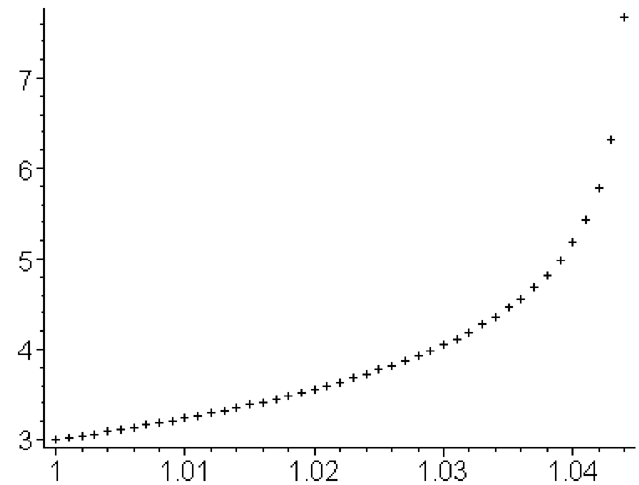
5. Write a Maple program to solve

$$\begin{aligned} \frac{dx}{dt} &= e^{xt} - \cos(x-t) \\ x(1) &= 3 \end{aligned}$$

Use fourth order Runge-Kutta formulas with $h = 0.001$. Stop the computation just before the solution overflows and graph the solution.

```
N := 1000; # hopefully a sufficient number of iterations
t[0] := 1.0;
x[0] := 3.0;
f := (t,x) -> exp(t*x)-cos(x-t);
dt := 0.001;
for i from 0 to N do
    t[i+1] := t[i] + dt;
    F1[i] := dt*f(t[i],x[i]);
    F2[i] := dt*f(t[i]+dt/2,x[i]+F1[i]/2);
    F3[i] := dt*f(t[i]+dt/2,x[i]+F2[i]/2);
    F4[i] := dt*f(t[i]+dt,x[i]+F3[i]);
    x[i+1] := x[i] + (F1[i] + 2*F2[i] + 2*F3[i] + F4[i])/6;
    if F2[i] > 10^6 then break fi;  # or else f(t,x) is getting too big to handle
od;
n := i-1;
datapoints := [seq([t[i],x[i]],i=0..n)]:
with(plots):
pointplot(datapoints);
```

This program produces the following graph:



6.* Derive the second order Runge-Kutta formula

$$x(t+h) = x(t) + hF\left(t + \frac{1}{2}h, x + \frac{1}{2}hF(t,x)\right)$$

by performing a Richardson Extrapolation on Euler's method using step sizes h and $h/2$. (*Hint*: assume the error term is Ch^2.)

7.* Derive the third order Runge-Kutta formulas

$$x(t+h) = x(t) + \frac{1}{9}(2F_1 + 3F_2 + 4F_3)$$

where

$$\begin{aligned} F_1 &= hF(t,x) \\ F_2 &= hF\left(t + \frac{1}{2}h, x + \frac{1}{2}F_1\right) \\ F_3 &= hF\left(t + \frac{3}{4}h, x + \frac{3}{4}F_2\right) \end{aligned}$$