Math 4513
Solutions to Homework 8

1. Write down the Richardson extrapolation for the derivative $f'(x)$ that is accurate to order 8 in h.

- In lecture we derived

$$
\begin{aligned}
f'(x) &= \phi_0(h) + \mathcal{O}(h^2) \\
&= \phi_1(h) + \mathcal{O}(h^4) \\
&= \phi_2(h) + \mathcal{O}\left(h^6\right) \\
&= \phi_3\left(h\right) + \mathcal{O}\left(h^8\right) \\
&= \cdots
\end{aligned}
$$

where

$$\phi_0 = \frac{f(x+h) - f(x-h)}{2}$$

and the higher Richardson extrapolations were given by the recursive formula.

$$\phi_n = \frac{4^n}{4^n - 1}\phi_{n-1}\left(\frac{h}{2}\right) - \frac{1}{4^n - 1}\phi_{n-1}(h)$$

We need to compute $\phi_3(h)$.

$$
\begin{aligned}
\phi_3(h) &= \frac{4^3}{4^3-1}\phi_2\left(h/2\right) - \frac{1}{4^3-1}\phi_2(h) \\
&= \frac{64}{65}\left(\frac{4^2}{4^2-1}\phi_1\left(\frac{h}{4}\right) - \frac{1}{4^2-1}\phi_1\left(\frac{h}{2}\right)\right) \\
&\quad -\frac{1}{65}\left(\frac{4^2}{4^2-1}\phi_1\left(\frac{h}{2}\right) - \frac{1}{4^2-1}\phi_1\left(h\right)\right) \\
&= \frac{(64)(16)}{(65)(15)}\left(\frac{4}{3}\phi_0\left(\frac{h}{8}\right) - \frac{1}{3}\phi_0\left(\frac{h}{4}\right)\right) \\
&\quad -\frac{(64)}{(65)(15)}\left(\frac{4}{3}\phi_0\left(\frac{h}{4}\right) - \frac{1}{3}\phi_0\left(\frac{h}{2}\right)\right) \\
&\quad -\frac{(16)}{(65)(15)}\left(\frac{4}{3}\phi_0\left(\frac{h}{4}\right) - \frac{1}{3}\phi_0\left(\frac{h}{2}\right)\right) \\
&\quad +\frac{1}{(65)(15)}\left(\frac{4}{3}\phi_0\left(\frac{h}{2}\right) - \frac{1}{3}\phi_0\left(h\right)\right) \\
&= \frac{2048}{2835}f(x+1/8\,h) - \frac{2048}{2835}f(x-1/8\,h) - \frac{32}{135}f(x+1/4\,h) + \frac{32}{135}f(x-1/4\,h) \\
&\quad +\frac{2}{135}f(x+1/2\,h) - \frac{2}{135}f(x-1/2\,h) - \frac{1}{5670}f(x+h) + \frac{1}{5670}f(x-h)
\end{aligned}
$$

This result could also be obtained with the following Maple program

```
F[0] := (f(x+h) - f(x-h))/2;
for i from 0 to 2 do
   F[i+1] := ((4^(i+1))*subs(h=h/2,F[i]) - F[i])/(4^(i+1) -1);
od:
F[3];
```

$\square$

2. Suppose

$$\int_a^b f(x)dx$$

is calculated numerically by interpolating the function $f(x)$ at the points

$$x_i = \frac{a+b}{2} + \frac{b-a}{2}\cos\left(\frac{(i+1)\pi}{n+2}\right) \quad , \quad i = 0,1,2,\ldots,n$$

and then integrating the interpolation polynomial between a and b. Express the maximal error in terms of a derivative of f, n, and the end points of integration a and b. (Hint: Write down a change of variables formula reduces the integral over $[a,b]$ to an integral over $[-1,1]$.)

- If $P(x)$ is a degree n polynomial interpolation of a function $f(x)$ then we have

$$|f(x) - P(x)| = \frac{1}{(n+1)!}f^{(n+1)}(\xi_x)\prod_{i=0}^{n}|x - x_i| \quad , \quad \forall\, x \in [a,b]$$

for some point $\xi_x \in [a,b]$. Thus, the error that occurs when we integrate $P(x)$ in place of $f(x)$ on $[a,b]$ will be

$$err = \int_a^b \left[\frac{1}{(n+1)!}f^{(n+1)}(\xi_x)\prod_{i=0}^{n}|x - x_i|\right] dx$$

In the case at hand, the interpolations nodes are the images of the zeros

$$t_i = \cos\left(\frac{(i+1)\pi}{n+2}\right)$$

of $U_{n+1}(t)$, the Chebyskev Polynomials of the Second Kind, by the linear mapping

$$\sigma : [-1,1] \to [a,b] \quad : \quad t \mapsto \frac{b+a}{2} + \left(\frac{b-a}{2}\right)t$$

We note that

$$\begin{aligned} \sigma^{-1}(a) &= -1 \\ \sigma^{-1}(b) &= 1 \\ x_i &= \sigma(t_i) \end{aligned}$$

Thus,

$$\begin{aligned} err &= \int_a^b \left[\frac{1}{(n+1)!}f^{(n+1)}(\xi_x)\prod_{i=0}^{n}|x - x_i|\right] dx \\ &= \frac{1}{(n+1)!}f^{(n+1)}(\xi_x)\int_a^b \prod_{i=0}^{n}|x - x_i|\, dx \\ &= \frac{1}{(n+1)!}f^{(n+1)}(\xi_x)\int_a^b \prod_{i=0}^{n}|x - \sigma(t_i)|\, dx \end{aligned}$$

The change of variables formula for 1-dimensional integrals is

$$\int_a^b F(x)\,dx = \int_{\sigma_{-1}(a)}^{\sigma^{-1}(b)} F(\sigma(t))\frac{d\sigma}{dt}dt$$

and so

$$\begin{aligned}
err &= \frac{1}{(n+1)!} f^{(n+1)}(\xi_x) \int_a^b \prod_{i=0}^{n} |x - \sigma(t_i)|\, dt \\
&= \frac{1}{(n+1)!} f^{(n+1)}(\xi_x) \int_{-1}^{1} \prod_{i=0}^{n} |\sigma(t) - \sigma(t_i)| \frac{d\sigma}{dt} dt \\
&= \frac{1}{(n+1)!} f^{(n+1)}(\xi_x) \int_{-1}^{1} \prod_{i=0}^{n} \left| \left(\frac{b+a}{2} + \frac{b-a}{2} t \right) - \left(\frac{b+a}{2} + \frac{b-a}{2} t_i \right) \right| \left(\frac{b-a}{2} \right) dx \\
&= \frac{1}{(n+1)!} f^{(n+1)}(\xi_x) \int_{-1}^{1} \prod_{i=0}^{n} \left| \frac{b-a}{2} t - \frac{b-a}{2} t_i \right| \left(\frac{b-a}{2} \right) dt \\
&= \frac{1}{(n+1)!} f^{(n+1)}(\xi_x) \left(\frac{b-a}{2} \right)^{n+2} \int_{-1}^{1} \prod_{i=0}^{n} |t - t_i|\, dt \\
&= \frac{1}{(n+1)!} f^{(n+1)}(\xi_x) \left(\frac{b-a}{2} \right)^{n+2} \left(2^{-n}\right)
\end{aligned}$$

In the last step we used the property that if the t_i are the zeros of $U_{n+1}(t)$, then

$$\int_{-1}^{1} \prod_{i=0}^{n} |t - t_i|\, dt = 2^{-n}$$

3. Find a quadrature formula for the integral

$$\int_a^{a+3h} f(x)$$

corresponding to the case where the function $f(x)$ is interpolated at four points: $x_0 = a$, $x_1 = a + h$, $x_2 = a + 2h$, and $x_3 = a + 3h$.

- Let $F(x)$ be the 4-point polynomial interpolation of $f(x)$ with nodes $x_0 = a$, $x_1 = a+h$, $x_2 = a+2h$, and $x_3 = a + 3h$. Recall that the Lagrange form of an interpolation polynomial is

$$F(x) = \sum_{i=0}^{n} f(x_i)\, \ell_i(x) \tag{0.1}$$

where the *cardinal functions* $\ell_i(x)$ are given by

$$\ell_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{(x - x_j)}{(x_i - x_j)}$$

We have

$$\begin{aligned}
\int_a^{a+3h} f(x)dx &\approx \int_a^{a+3h} F(x)dx \\
&= \int_a^{a+3h} \sum_{i=0}^{3} f(x_i)\ell_i(x)dx \\
&= \sum_{i=0}^{3} f(x_i) \int_a^{a+3h} \ell_i(x)dx \\
&= \sum_{i=0}^{3} f(x_i) A_i
\end{aligned}$$

where the constants A_i are the integrals of the cardinal functions $\ell_i(x)$. Since equation (0.1) must be exact whenever $f(x)$ is a polynomial of degree less than or equal to n (because then the interpolation polynomial must coincide with $f(x)$), we have takiing $f(x) = x^k$, $k = 0, 1, 2, 3$,

$$\begin{aligned}\frac{1}{k+1}(a+3h)^{k+1} - \frac{1}{k+1}a^{k+1} &= \int_a^{a+3h} x^k dx = \sum_{i=0}^{3} f(x_i)A_i = x_0^k A_0 + x_1^k A_1 + x_2^k A_2 + x_3^k A_3 \\ &= a^k A_0 + (a+h)^k A_1 + (a+2h)^k A_2 + (a+3h)^k A_3\end{aligned}$$

or

$$\begin{aligned}A_0 + A_1 + A_2 + A_3 &= 3h \\ a^k A_0 + (a+h)^k A_1 + (a+2h)^k A_2 + (a+3h)^k A_3 &= \frac{1}{2}(a+3h)^2 - \frac{1}{2}a^2 \\ a^k A_0 + (a+h)^k A_1 + (a+2h)^k A_2 + (a+3h)^k A_3 &= \frac{1}{3}(a+3h)^3 - \frac{1}{3}a^3 \\ a^k A_0 + (a+h)^k A_1 + (a+2h)^k A_2 + (a+3h)^k A_3 &= \frac{1}{3}(a+3h)^{k+1} - \frac{1}{3}a^{k+1}\end{aligned}$$

Solving this set of equations for A_0, A_1, A_2, and A_3 yields

$$\begin{aligned}A_0 &= \frac{3}{8}h \\ A_1 &= \frac{9}{8}h \\ A_2 &= \frac{9}{8}h \\ A_3 &= \frac{3}{8}h\end{aligned}$$

Thus,

$$\int_a^{a+3h} f(x) \approx \frac{h}{8}\left(3f(a) + 9f(a+h) + 9f(a+2h) + 3f(a+3h)\right)$$

4. (a) Write a Maple program that applies the formula

$$\begin{aligned}\int_a^b f(x)dx &\approx \sum_{i=0}^{n-1} f\left(\tilde{x}_i\right)\Delta x \\ \Delta x &= \frac{b-a}{n} \quad , \quad \tilde{x}_i = a + \Delta x i + \frac{\Delta x}{2} \quad , \quad \text{the midpoint of the interval } [x_i, x_{i+1}]\end{aligned}$$

to calculate

$$\int_0^2 e^{x^2} dx$$

(take n, the number of subdivsions of $[0, 2]$ equal to 20).

- From the formal definition of the Riemann integral we have

$$\int_a^b f(x)dx \approx \sum_{i=0}^{n-1} f\left(\tilde{x}_i\right)\Delta x$$

where

$$\Delta x = \frac{b-a}{n} \quad , \quad \tilde{x}_i = a + \Delta x i + \frac{\Delta x}{2}$$

This formula leads to the following Maple code.

```
f := x -> evalf(exp(x^2));
a := 0.0;
b := 2.0;
n := 20;
I1 := 0.0;
```

```
dx := (b-a)/n;
x := a + dx/2;
for i from 0 to n-1 do
  I1 := I1 + f(x)*dx;
  x := x+dx;
od;
I1;
```

```
Output: I1 = 16.36221034
```

(b) Use the result of Problem 3 to formulate a numerical recipe for calculating

$$\int_a^b f(x)\,dx$$

and then write a Maple program that computes

$$\int_0^2 e^{x^2}\,dx$$

(take n, the number of subdivsions of $[0,2]$ equal to 20).

- We have

$$\int_a^b f(x)dx = \sum_{i=0}^{n-1} \int_{x_i}^{x_i+\Delta x} f(x)dx$$

We shall use the results of Problem 3 to get fairly accurate approximation for the summands

$$\int_{x_i}^{x_i+\Delta x} f(x)dx \quad .$$

Setting

$$\Delta x = 3h \quad \Rightarrow \quad h = \frac{\Delta x}{3}$$

we obtain from the result of Problem 3

$$\int_{x_i}^{x_i+\Delta x} f(x)dx \approx \frac{\Delta x}{(3)(8)}\left(3f(x_i) + 9f(x_i+\frac{\Delta x}{3}) + 9f(x_i+\frac{2\Delta x}{3}) + 3f(x_i+\Delta x)\right)$$

Thus,

$$\int_a^b f(x)dx = \sum_{i=0}^{n-1}\frac{\Delta x}{(3)(8)}\left(3f(x_i) + 9f(x_i+\frac{\Delta x}{3}) + 9f(x_i+\frac{2\Delta x}{3}) + 3f(x_i+\Delta x)\right)$$

This formula leads to the following Maple code.

```
f := x -> evalf(exp(x^2));
a := 0.0;
b := 2.0;
n := 20;
I1 := 0.0;
dx := (b-a)/n;
x := a
for i from 0 to n-1 do
  DI1 := dx*(3*f(x) + 9*f(x+dx/3) + 9*f(x+2*dx/3) + 3*f(x+dx))/24;
  I1 := I1 + DI1;
  x := x+dx;
```

```
od:
I1;


Output I = 16.45270135
```