Math 4513
Solutions to Homework 6

1. Find the Newton and Lagrange forms for the interpolation polynomial corresponding to the following sets of data.

(a)

$$\begin{array}{ll} x_0 = 0 & y_0 = -1 \\ x_1 = 1 & y_1 = -2 \\ x_2 = 2 & y_2 = -1 \\ x_3 = 3 & y_3 = -4 \end{array}$$

- The Newton form of the interpolation polynomial is computed as follows:

$$\begin{aligned} c_0 &= y_0 = -1 \\ P_0(x) &= c_0 \\ c_1 &= \frac{y_1 - P_0(x_1)}{(x_1 - x_0)} = \frac{-2+1}{(1-0)} = -1 \\ P_1(x) &= P_0(x) + c_1(x - x_0) = -1 - (x - 0) = -1 - x \\ c_2 &= \frac{y_2 - P_1(x_2)}{(x_2 - x_0)(x_2 - x_1)} = \frac{-1+3}{(2)(1)} = 1 \\ P_2(x) &= P_1(x) + c_2(x - x_0)(x - x_1) = -1 - x + x(x-1) \\ c_3 &= \frac{y_3 - P_2(x_3)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)} = \frac{-4-(2)}{(3)(2)(1)} = -1 \end{aligned}$$

So

$$\begin{aligned} P(x) &= c_0 + c_1(x - x_0) + c_2(x - x_0)(x - x_1) + c_3(x - x_0)(x - x_1)(x - x_2) \\ &= -1 - x + x(x-1) - x(x-1)(x-2) \end{aligned}$$

- The Lagrange form the interpolation polynomial is given by

$$P(x) = \sum_{i=0}^{n} y_i \left({}_i \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{(x - x_j)}{(x_i - x_j)} \right)$$

In our case, $n = 3$ and

$$\begin{aligned} P(x) &= y_0 \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} + y_1 \frac{(x - x_0)(x - x_2)(x - x_3)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)} \\ &\quad + y_2 \frac{(x - x_0)(x - x_1)(x - x_3)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)} + y_3 \frac{(x - x_0)(x - x_1)(x - x_2)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)} \\ &= (-1)\frac{(x-1)(x-2)(x-3)}{(-1)(-2)(-3)} + (-2)\frac{x(x-2)(x-3)}{(1)(-1)(-2)} \\ &\quad + (-1)\frac{x(x-1)(x-3)}{(2)(1)(-1)} + (-4)\frac{x(x-1)(x-2)}{(3)(2)(1)} \\ &= \frac{1}{6}(x-1)(x-2)(x-3) - x(x-2)(x-3) + \frac{1}{2}x(x-1)(x-3) - \frac{2}{3}x(x-1)(x-2) \end{aligned}$$

$\square$

(b)

$$\begin{array}{ll} x_0 = 1 & y_0 = 3 \\ x_1 = 2 & y_1 = 2 \\ x_2 = 0 & y_2 = -4 \\ x_3 = 3 & y_3 = 5 \end{array}$$

- Carrying out the calculation as in Part (a), we have

$$\begin{aligned}
c_0 &= y_0 = 3 \\
P_0(x) &= c_0 = 3 \\
c_1 &= \frac{y_1 - P_0(x_1)}{(x_1 - x_0)} = \frac{2-3}{(2-1)} = -1 \\
P_1(x) &= P_0(x) + c_1(x - x_0) = 3 - (x-1) \\
c_2 &= \frac{y_2 - P_1(x_2)}{(x_2 - x_0)(x_2 - x_1)} = \frac{-4-(4)}{(0-1)(0-2)} = -4 \\
P_2(x) &= P_1(x) + c_2(x - x_0)(x - x_1) = 3 - (x-1) - 4(x-1)(x-2) \\
c_3 &= \frac{y_3 - P_2(x_3)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)} = \frac{5-(11)}{(3)(2)(1)} = -1
\end{aligned}$$

so the Newton form of the interpolation polynomial is

$$P(x) = 3 - (x-1) - 4(x-1)(x-2) - (x-1)(x-2)(x)$$

- The Lagrange form of the interpolation polynomial is

$$\begin{aligned}
P(x) &= y_0 \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} + y_1 \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \\
&\quad + y_2 \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} + y_3 \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} \\
&= 3\frac{(x-2)(x)(x-3)}{(-1)(1)(-2)} + (2)\frac{(x-1)(x)(x-3)}{(1)(2)(-1)} \\
&\quad + (-4)\frac{(x-1)(x-2)(x-3)}{(1)(2)(3)} + (5)\frac{(x-1)(x-2)(x)}{(2)(1)(3)} \\
&= \frac{3}{2}x(x-2)(x-3) - x(x-1)(x-3) - \frac{2}{3}(x-1)(x-2)(x-3) + \frac{5}{6}(x)(x-1)(x-2)
\end{aligned}$$

□

2. What is the maximal error that can occur in approximating $f(x) = \cosh(x)$ by a polynomial interpolation at 6 points in the interval $[0, -1]$.

- According to the theorem of section 6.1, the error term for an $n+1$ point polynomial interpolation of a function on an interval $[a, b]$ is

$$E = \frac{1}{(n+1)!} f^{(n+1)}(\xi_x) \prod_{i=0}^{n} (x - x_i) \quad , \quad \text{for some } \xi_x \in [a, b]$$

In the case at hand, $f(x) = \cosh(x)$, $n = 5$, $[a, b] = [0, 1]$ and

$$f^{(6)}(x) = \cosh(x)$$

Thus, the maximum possilbe value of the factor $f^{(n+1)}(\xi_x)$ will be

$$\max_{x \in [0,1]} \cosh(x) = \cosh(1) = 1.543 \ldots$$

The nodes x_i must also in $[0,1]$, and the so the maximal possible value of a factor $(x - x_i)$ in the interval would be 1. Hence

$$E \leq \frac{1}{6!}(1.544) \times \prod_{i=0}^{n}(1) = 0.00214$$

$\square$

3. Suppose you had to design an experiment that would determine an interpolating polynomial for a function that takes values in the range between 1 and 100. If you can only take 10 data points, which points should you choose?

- The optimal set of 10 nodes for the interval $[-1,1]$ are the roots of Chebyshev polynomial $T_{10}(x)$. These are

$$\tilde{x}_i = \cos\left(\frac{2i+1}{20}\pi\right) \quad , \quad i = 0, 1, \ldots, 9$$

The optimal set of 10 nodes on the interval $[1,100]$ can be obtained by simply mapping the interval $[-1,1]$ (and hence the points x_i) linearly onto the interval $[1,100]$. We thus need a linear function that takes -1 to 1 and 1 to 100. It is easy to check that

$$f(x) = \frac{101}{2} + \frac{99}{2}x$$

is the unique linear function such that $f(-1) = 1$ and $f(1) = 100$. The optimal choice of zeros for the interval $[1,100]$ is therefore

$$x_i = f(\tilde{x}_i) = \frac{101}{2} + \frac{99}{2}\cos\left(\frac{2i+1}{2n+2}\pi\right), \quad i = 0, 1, \ldots, 9$$

These are

$$\begin{aligned}
x_0 &= 99.39057286 \\
x_1 &= 94.60482295 \\
x_2 &= 85.50178566 \\
x_3 &= 72.97252973 \\
x_4 &= 58.24350603 \\
x_5 &= 42.75649397 \\
x_6 &= 28.02747027 \\
x_7 &= 15.49821434 \\
x_8 &= 6.39517705 \\
x_9 &= 1.60942714
\end{aligned}$$

$\square$