MATH 4513 : HOMEWORK SOLUTIONS 5

1. Solve the following linear systems twice. First use Gaussian elimination and give the factorization $\mathbf{A} = \mathbf{LU}$. Second, use Gaussian elimination with scaled pivoting and determine the factorization of the form $\mathbf{PA} = \mathbf{LU}$.

(a)

$$\begin{pmatrix} -1 & 1 & -4 \\ 2 & 2 & 0 \\ 3 & 3 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ \frac{1}{2} \end{pmatrix}$$

- **Standard Gaussian Elimination:** The augmented matrix is

$$\begin{pmatrix} -1 & 1 & -4 & 0 \\ 2 & 2 & 0 & 1 \\ 3 & 3 & 2 & \frac{1}{2} \end{pmatrix}$$

After the first stage of Gaussian elmination we have

$$\begin{pmatrix} -1 & 1 & -4 & 0 \\ \underline{-2} & 4 & -8 & 1 \\ \underline{-3} & 6 & -10 & \frac{1}{2} \end{pmatrix}$$

After the second (and final stage) of Gaussian elimination we have

$$\begin{pmatrix} -1 & 1 & -4 & 0 \\ \underline{-2} & 4 & -8 & 1 \\ \underline{-3} & \underline{\frac{3}{2}} & 2 & -1 \end{pmatrix}$$

The LU factorization of $\mathbf{A}$ is thus given by

$$\mathbf{L} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & \frac{3}{2} & 1 \end{pmatrix} \quad , \quad \mathbf{U} = \begin{pmatrix} -1 & 1 & -4 \\ 0 & 4 & -8 \\ 0 & 0 & 2 \end{pmatrix}$$

And the original linear system is equivalent to

$$\begin{pmatrix} -1 & 1 & -4 \\ 0 & 4 & -8 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

This system is easily solved by back substitution. The result is

$$\begin{aligned} x_3 &= -\frac{1}{2} \\ x_2 &= -\frac{3}{4} \\ x_1 &= \frac{5}{4} \end{aligned}$$

$\square$

- **Gaussian Elimination with Scaled Pivoting:** To begin with we have

$$\mathbf{A} = \begin{pmatrix} -1 & 1 & -4 \\ 2 & 2 & 0 \\ 3 & 3 & 2 \end{pmatrix}$$

We also initialize the permutation vector $\mathbf{p} = (1,2,3)$ (which just means that we haven't, as yet, interchanged any rows). Now we compute the qualities of each of the three rows of $\mathbf{A}$:

$$\begin{aligned} q_1 &= \frac{|a_{11}|}{\max\{|a_{11}|,|a_{12}|,|a_{13}|\}} = \frac{1}{4} \\ q_2 &= \frac{|a_{21}|}{\max\{|a_{21}|,|a_{22}|,|a_{23}|\}} = \frac{1}{1} = 1 \\ q_3 &= \frac{|a_{11}|}{\max\{|a_{31}|,|a_{32}|,|a_{33}|\}} = \frac{3}{3} = 1 \end{aligned}$$

The highest qualities are achieved by rows 2 and 3, and so we could use either of these two rows as the pivoting row for the first stage of Gaussian elimination. We'll choose row 2. Interchanging row 2 with row 1 we have

$$\mathbf{A} \quad \Rightarrow \quad \begin{pmatrix} 2 & 2 & 0 \\ -1 & 1 & -4 \\ 3 & 3 & 2 \end{pmatrix} \qquad , \qquad \mathbf{p} \quad \Rightarrow \quad (2,1,3)$$

We now pivot about the first row

$$\Rightarrow \quad \begin{pmatrix} 2 & 2 & 0 \\ \underline{-\frac{1}{2}} & 2 & -4 \\ \underline{\frac{3}{2}} & 0 & 2 \end{pmatrix}$$

Now at this stage we're actually through with Gaussian elimination. However, since we also need the corresponding LU factorization, we still need to identify the mulitplier that goes in the 32 slot of the matrix $\mathbf{L}$. This will just be a 0 since that's the factor we need to multiply the second row by to get the third row in the right form. Therefore, we end up with

$$\begin{pmatrix} 2 & 2 & 0 \\ \underline{-\frac{1}{2}} & 2 & -4 \\ \underline{\frac{3}{2}} & \underline{0} & 2 \end{pmatrix}$$

$$\Rightarrow \quad \mathbf{L} = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ \frac{3}{2} & 0 & 1 \end{pmatrix} \qquad , \qquad \mathbf{U} = \begin{pmatrix} 2 & 2 & 0 \\ 0 & 2 & -4 \\ 0 & 0 & 2 \end{pmatrix}$$

and our final permutation vector is

$$\mathbf{p} = (2,1,3) \quad .$$

We now need to solve

$$\mathbf{L}(\mathbf{Ux}) = (\mathbf{PA})\,\mathbf{x} = \mathbf{Pb} = \begin{pmatrix} 1 \\ 0 \\ \frac{1}{2} \end{pmatrix}$$

We do this in two stages. First we solve $\mathbf{Lz} = \mathbf{b}$ for $\mathbf{z}$.

$$\begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ \frac{3}{2} & 0 & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \frac{1}{2} \end{pmatrix}$$

leads to a system of equations that we can solve by forward substitution

$$\left.\begin{aligned} z_1 &= 1 \\ -\tfrac{1}{2}z_1 + z_2 &= 0 \\ \tfrac{3}{2}z_1 + z_3 &= \tfrac{1}{2} \end{aligned}\right\} \quad \Rightarrow \left\{\begin{aligned} z_1 &= 1 \\ z_2 &= \tfrac{1}{2} \\ z_3 &= -1 \end{aligned}\right.$$

Now we solve $\mathbf{U}\mathbf{x} = \mathbf{z}$

$$\begin{pmatrix} 2 & 2 & 0 \\ 0 & 2 & -4 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1}{2} \\ -1 \end{pmatrix}$$

or

$$\left.\begin{array}{rcl} 2x_3 & = & -1 \\ 2x_2 - 4x_3 & = & \frac{1}{2} \\ 2x_1 + 2x_2 & = & 1 \end{array}\right\} \Rightarrow \left\{\begin{array}{rcl} x_1 & = & -\frac{1}{4} \\ x_2 & = & -\frac{3}{4} \\ x_3 & = & -\frac{1}{2} \end{array}\right.$$

$\square$

(b)

$$\begin{pmatrix} 1 & 6 & 0 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$$

- **Standard Gaussian Elimination:** The augmented matrix is

$$\begin{pmatrix} 1 & 6 & 0 & 3 \\ 2 & 1 & 0 & 1 \\ 0 & 2 & 1 & 1 \end{pmatrix}$$

After the first stage of Gaussian elmination we have

$$\begin{pmatrix} 1 & 6 & 0 & 3 \\ \underline{2} & -11 & 0 & -5 \\ \underline{0} & 2 & 1 & 1 \end{pmatrix}$$

After the second (and final stage) of Gaussian elimination we have

$$\begin{pmatrix} 1 & 6 & 0 & 3 \\ \underline{2} & -11 & 0 & -5 \\ \underline{0} & \underline{-\frac{1}{11}} & 1 & \frac{1}{11} \end{pmatrix}$$

The LU factorization of $\mathbf{A}$ is thus given by

$$\mathbf{L} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & -\frac{1}{11} & 1 \end{pmatrix} \quad , \quad \mathbf{U} = \begin{pmatrix} 1 & 6 & 0 \\ 0 & -11 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

And the original linear system is equivalent to

$$\begin{pmatrix} 1 & 6 & 0 \\ 0 & -11 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ -5 \\ \frac{1}{11} \end{pmatrix}$$

This system is easily solved by back substitution. The result is

$$\begin{aligned} x_3 &= \frac{1}{11} \\ x_2 &= \frac{5}{11} \\ x_1 &= \frac{3}{11} \end{aligned}$$

$\square$

- **Gaussian Elimination with Scaled Pivoting:** To begin with we have

$$\mathbf{A} = \begin{pmatrix} 1 & 6 & 0 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}$$

We also initialize the permutation vector $\mathbf{p} = (1,2,3)$ (which just means that we haven't, as yet, interchanged any rows). Now we compute the qualities of each of the three rows of $\mathbf{A}$:

$$\begin{aligned} q_1 &= \frac{|a_{11}|}{\max\{|a_{11}|,|a_{12}|,|a_{13}|\}} = \frac{1}{6} \\ q_2 &= \frac{|a_{21}|}{\max\{|a_{21}|,|a_{22}|,|a_{23}|\}} = \frac{2}{2} = 1 \\ q_3 &= \frac{|a_{11}|}{\max\{|a_{31}|,|a_{32}|,|a_{33}|\}} = \frac{0}{3} = 0 \end{aligned}$$

The highest qualities are achieved by rows 2 and so we interchanging row 2 with row 1 :

$$\mathbf{A} \quad \Rightarrow \quad \begin{pmatrix} 2 & 1 & 0 \\ 1 & 6 & 0 \\ 0 & 2 & 1 \end{pmatrix} \qquad , \qquad \mathbf{p} \quad \Rightarrow \quad (2,1,3)$$

We now pivot about the first row

$$\Rightarrow \begin{pmatrix} 2 & 1 & 0 \\ \underline{\frac{1}{2}} & \frac{11}{2} & 0 \\ \underline{0} & 2 & 1 \end{pmatrix}$$

We now compare the qualities of the last two rows:

$$\begin{aligned} q_2 &= \frac{\frac{11}{2}}{\frac{11}{2}} = 1 \\ q_3 &= \frac{2}{2} = 1 \end{aligned}$$

They both have quality 1 so we won't bother exchanging rows. Carrying out the final stage of Gaussian elimination now yields

$$\begin{pmatrix} 2 & 1 & 0 \\ \underline{\frac{1}{2}} & \frac{11}{2} & 0 \\ \underline{0} & \underline{\frac{4}{11}} & 1 \end{pmatrix}$$

Now we're done with Gaussian elimination. Our LU factorization is

$$\mathbf{L} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & \frac{4}{11} & 1 \end{pmatrix} \qquad , \qquad \mathbf{U} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & \frac{11}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and our final permutation vector is

$$\mathbf{p} = (2,1,3) \quad .$$

We now need to solve

$$\mathbf{L}(\mathbf{Ux}) = (\mathbf{PA})\,\mathbf{x} = \mathbf{Pb} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$$

We do this in two stages. First we solve $\mathbf{Lz} = \mathbf{b}$ for $\mathbf{z}$.

$$\begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & \frac{4}{11} & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$$

leads to a system of equations that we can solve by forward substitution

$$\left.\begin{aligned} z_1 &= 1 \\ \tfrac{1}{2}z_1 + z_2 &= 3 \\ \tfrac{4}{11}z_2 + z_3 &= 1 \end{aligned}\right\} \quad \Rightarrow \left\{\begin{aligned} z_1 &= 1 \\ z_2 &= \tfrac{5}{2} \\ z_3 &= \tfrac{1}{11} \end{aligned}\right.$$

5

Now we solve $\mathbf{Ux} = \mathbf{z}$

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & \frac{11}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{5}{2} \\ \frac{1}{11} \end{pmatrix}$$

or

$$\left.\begin{array}{rcl} 2x_3 & = & -1 \\ 2x_2 - 4x_3 & = & \frac{1}{2} \\ 2x_1 + 2x_2 & = & 1 \end{array}\right\} \Rightarrow \left\{\begin{array}{rcl} x_1 & = & \frac{3}{11} \\ x_2 & = & \frac{5}{11} \\ x_3 & = & \frac{1}{11} \end{array}\right.$$

□

2. Show how Gaussian elimination with scaled row pivoting works on these examples (LU factorization phase only).

(a)

$$\mathbf{A} = \begin{pmatrix} 2 & -2 & 4 \\ 1 & 1 & -1 \\ 3 & 7 & 5 \end{pmatrix}$$

- We have

  $$q_1 = \frac{1}{2} \quad , \quad q_2 = 1 \quad , \quad q_3 = \frac{3}{7}$$

  So we'll interchange the first two rows

  $$\mathbf{p} \quad \Rightarrow \quad (2,1,3) \qquad , \qquad \mathbf{A} \quad \Rightarrow \quad \begin{pmatrix} 1 & 1 & -1 \\ 2 & -2 & 4 \\ 3 & 7 & 5 \end{pmatrix}$$

  After pivoting about the first row we have

  $$\begin{pmatrix} 1 & 1 & -1 \\ \underline{2} & -4 & 6 \\ \underline{3} & 4 & 8 \end{pmatrix}$$

  We now have

  $$q_2 = \frac{2}{3} \quad , \quad q_3 = \frac{1}{2}$$

  Since $q_2 > q_3$ we won't bother interchanging rows. Carrying out the next stage of Gaussian elimination yields

  $$\begin{pmatrix} 1 & 1 & -1 \\ \underline{2} & -4 & 6 \\ \underline{3} & \underline{-1} & 14 \end{pmatrix}$$

  Hence,

  $$\mathbf{L} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & -1 & 1 \end{pmatrix} \quad , \quad \mathbf{U} = \begin{pmatrix} 1 & 1 & -1 \\ 0 & -4 & 6 \\ 0 & 0 & 14 \end{pmatrix}$$

  and

  $$\mathbf{p} = (2,1,3)$$

□

(b)

$$\mathbf{A} = \begin{pmatrix} 3 & 7 & 3 \\ 1 & \frac{7}{3} & 4 \\ 4 & \frac{4}{3} & 0 \end{pmatrix}$$

- We have

$$q_1 = \frac{3}{7} \quad , \quad q_2 = \frac{1}{4} \quad , \quad q_3 = 1$$

So we'll interchange the first and last rows

$$\mathbf{p} \quad \Rightarrow \quad (3,2,1) \qquad , \qquad \mathbf{A} \quad \Rightarrow \quad \begin{pmatrix} 4 & \frac{4}{3} & 0 \\ 1 & \frac{7}{3} & 4 \\ 3 & 7 & 3 \end{pmatrix}$$

After pivoting about the first row we have

$$\begin{pmatrix} 4 & \frac{4}{3} & 0 \\ \underline{\frac{1}{4}} & 2 & 4 \\ \underline{\frac{3}{4}} & 6 & 3 \end{pmatrix}$$

We now have

$$q_2 = \frac{1}{2} \quad , \quad q_3 = 1$$

Since $q_3 > q_2$, we'll interchanging rows.

$$\mathbf{p} \quad \Rightarrow \quad (3,1,2) \qquad , \qquad \begin{pmatrix} 4 & \frac{4}{3} & 0 \\ \underline{\frac{3}{4}} & 6 & 3 \\ \underline{\frac{1}{4}} & 2 & 4 \end{pmatrix}$$

Carrying out the next stage of Gaussian elimination yields

$$\begin{pmatrix} 4 & \frac{4}{3} & 0 \\ \underline{\frac{3}{4}} & 6 & 3 \\ \underline{\frac{1}{4}} & \underline{\frac{1}{3}} & 3 \end{pmatrix}$$

Hence,

$$\mathbf{L} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{3}{4} & 1 & 0 \\ \frac{1}{4} & \frac{1}{3} & 1 \end{pmatrix} \quad , \quad \mathbf{U} = \begin{pmatrix} 4 & \frac{4}{3} & 0 \\ 0 & 6 & 3 \\ 0 & 0 & 3 \end{pmatrix}$$

and

$$\mathbf{p} = (3,1,2)$$

□