MATH 4513 : HOMEWORK 3

1. Show that

$$e_{n+1} \approx \frac{|f''(r)|}{2\,|f'(r)|} e_n e_{n-1} = C e_n e_{n-1}$$

- Let r be the actual root of $f(x) = 0$, let x_n be the approximate value for r obtained by carrying out n iterations of the secant method, and let e_n be the corresponding error:

$$e_n = x_n - r.$$

We then have

$$\begin{aligned}
e_{n+1} &= x_{n+1} - r \\
&= x_n - f(x_n)\left(\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}\right) - r \\
&= e_n - f(x_n)\left(\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}\right) \\
&= e_n - f(x_n)\left(\frac{x_n - r - x_{n-1} + r}{f(x_n) - f(x_{n-1})}\right) \\
&= e_n - f(x_n)\left(\frac{e_n - e_{n-1}}{f(x_n) - f(x_{n-1})}\right) \\
&= \frac{e_n\left(f(x_n) - f(x_{n-1})\right) - f(x_n)\left(e_n - e_{n-1}\right)}{f(x_n) - f(x_{n-1})} \\
&= \frac{e_n f(x_{n-1}) - e_{n-1} f(x_n)}{f(x_n) - f(x_{n-1})} \\
&= \left[\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}\right]\left[\frac{e_n f(x_{n-1}) - e_{n-1} f(x_n)}{x_n - x_{n-1}}\right] \\
&= \left[\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}\right]\left[\frac{f(x_{n-1})/e_{n-1} - f(x_n)/e_n}{x_n - x_{n-1}}\right] e_n e_{n-1}
\end{aligned}$$

Now by Taylor's Theorem

$$\begin{aligned}
f(x_n) &= f(r) + f'(r)e_n + \frac{1}{2}f''(r)e_n^2 + \mathcal{O}\left(e_n^3\right) \\
&= 0 + f'(r)e_n + \frac{1}{2}f''(r)e_n^2 + \mathcal{O}\left(e_n^3\right)
\end{aligned}$$

So

$$\frac{f(x_n)}{e_n} = f'(r) + \frac{1}{2}f''(r)e_n + \mathcal{O}\left(e_n^2\right)$$

and similarly

$$\frac{f(x_{n-1})}{e_{n-1}} = f'(r) + \frac{1}{2}f''(r)e_{n-1} + \mathcal{O}\left(e_{n-1}^2\right)$$

So we have

$$\begin{aligned}\frac{f(x_n)}{e_n} - \frac{f(x_{n-1})}{e_{n-1}} &= \left(f'(r) + \frac{1}{2}f''(r)e_n\right) - \left(f'(r) - \frac{1}{2}f''(r)e_{n-1}\right) + \mathcal{O}\left(e_{n-1}^2\right) \\ &= \frac{1}{2}f''(r)\left(e_n - e_{n-1}\right) + \mathcal{O}\left(e_{n-1}^2\right)\end{aligned}$$

and

$$e_{n+1} \approx \left[\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}\right]\left[\frac{\frac{1}{2}f''(r)(e_n - e_{n-1})}{x_n - x_{n-1}}\right] e_n e_{n-1}$$

Now

$$e_n - e_{n-1} = (x_n - r) - (x_{n-1} - r) = x_n - x_{n-1}$$

and for x_n and x_{n-1} sufficiently close to r

$$\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} \approx f'(r)$$

So

$$e_{n+1} \approx [f'(r)]\left[\frac{1}{2}f''(r)\right] e_n e_{n-1} = C e_n e_{n-1} \tag{3.1}$$

□

2. Use the secant method to find a solution of

$$exp(x^2 - 2) = 3ln(x)$$

starting with $x0 = 1.5$, $x1 = 1.4$.

- The following Maple program will carry out the calculation

```
M := 10;
delta := 0.000001;
epsilon := 0.000001;
f := x -> exp(x^2 - 2) - 3*ln(x);
x0 := 1.5;
x1 := 1.4;
for i from 1 to M do
    x2 := x1 - f(x1)*(x1 -x0)/(f(x1) -f(x0));
    if (abs(f(x2)) < epsilon) then break; fi;
    x0 := x1;
    x1 := x2;
    if (abs(x1 -x0) < delta) then  break; fi;
    od:
x2;
f(x2);
```

The output of this program is

$$\begin{aligned} x2 &= 1.455716732 \\ f(x2) &= 0.46 \times 10^{-7}\end{aligned}$$

□