MATH 4513 : HOMEWORK 2

1. Let $[a_n, b_n]$ denote the sucessive intervals that arise from applying the bisection method to find a root of a continuous function $f(x)$. Let $c_n = \frac{1}{2}(a_n + b_n)$, $r = \lim_{n\to\infty} c_n$, and $e_n = r - c_n$.

(a) Show that $|e_n| \leq 2^{-n} |b-a|$.

- At the start of the bisection method, it is only know that a zero lies somewhere between a and b. After the first step, the zero is trapped either between a and $\frac{a+b}{2}$, or between $\frac{a+b}{2}$ and b. In either case the zero is trapped within a interval of length $\frac{|a-b|}{2}$. Each successive iteration of the bisection method will isolate the zero within an interval that is half the size of the interval in the preceding step. Hence the length of the interval after n iterations of the bisection method will be
$$\frac{|b-a|}{2^n}$$
If we take c_n to be the midpoint of the interval after n iterations, then the difference between c_n and the actual root can be at most half the length of the corresponding interval. Hence
$$|e_n| \leq 2^{-n} |b-a|$$

$\square$

(b) Show that $e_n = \mathcal{O}(2^{-n})$ as $n \to \infty$.

- From part (a) we see that by taking $C = |b-a|$, we have for all n
$$|e_n| \leq C \left|2^{-n}\right|$$
Hence
$$e_n = \mathcal{O}\left(n^{-n}\right)$$

$\square$

2. Using the bisection algorithm, find a root of $f(x) = x - \tan(x)$ in the interval $[1,2]$.

```
f := x -> x - tan(x);
a := 1.0;
b := 2.0;
c := (a+b)/2;
for i from 1 to 1000 do
     if (f(c) < 0) then
          a := c;
     else
          b := c:
     fi;
     c := (a+b)/2:
     if abs(f(c)) < 0.00001 then break fi:
     if abs(a-b) < 0.00001 then break fi:
od:
c;
f(c);
```

Output:

$$\begin{aligned} c &= 1.570796968 \\ f(c) &= -0.1559564945 \times 10^{6} \end{aligned}$$

The answer is *extremely wrong*. The reason why the bisection algorithm fails for this problem can be traced to the fact that the function $f(x)$ is discontinuous at $x = \pi/2 \approx 1.570796\ldots$.

$\square$

3. Using the bisection algorithm, find a root of $f(x) = 2^{-x} + e^{x} + 2\cos(x) - 6$ on $[1, 3]$.

```
f := x -> 2^(-x) + exp(x) + 2*cos(x) -6;
a := 1.0;
b := 3.0;
c := (a+b)/2;
for i from 1 to 1000 do
     if (f(c) < 0) then
          a := c;
     else
          b := c:
     fi;
     c := (a+b)/2:
     if abs(f(c)) < 0.00001 then break fi:
     if abs(a-b) < 0.00001 then break fi:
od:
c;
```

Output:

$$\begin{aligned} c &= 1.829383852 \\ f(c) &= 0.1026 \times 10^{-5} \end{aligned}$$

$\square$

4. With a hand held calculator, perform four iterations of Newton's Method to find an approximate zero of

$$f(x) = 4x^3 - 2x^2 + 3$$

starting with $x_0 = -1$.

- We have

  $$f'(x) = 12x^2 - 4x$$

  and so the Newton's method algorithm is given explicitly by

  $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{4x_n^3 - 2x_n^2 + 3}{12x^2 - 4x}$$

  Taking $x_0 = -1$ we have

  $$\begin{aligned} x_1 &= -0.8125000000 \\ x_2 &= -0.7708041959 \\ x_3 &= -0.7688323842 \\ x_4 &= -0.7688280859 \end{aligned}$$

$\square$

5. Write a computer program to solve

$$x = \tan x$$

by means of Newton's method. Find the roots nearest 4.5 and 7.7.

- The following two Maple programs will do the trick.

```
f := x -> x - tan(x);
f1 := x -> 1 - (sec(x))^2;
x0 := 4.5;
for i from 1 to 100 while ( abs(f(x0)) > 0.000001) do
x0 := x0 - f(x0)/f1(x0);
od;
x0;
f(x0);
```

Output:

$$\begin{aligned} x0 &= 4.493409458 \\ f(x0) &= 0.2 \times 10^{-9} \end{aligned}$$

```
f := x -> x - tan(x);
f1 := x -> 1- (sec(x))^2;
x0 := 7.7;
for i from 1 to 100 while ( abs(f(x0)) > 0.000001) do
x0 := x0 - f(x0)/f1(x0);
od;
x0;
f(x0);
```

Output:

$$\begin{aligned} x0 &= 7.725251837 \\ f(x0) &= -4.0 \times 10^{-9} \end{aligned}$$

□

6. Write a computer program to solve

$$x^3 + 3x = 5x^2 + 7$$

by Newton's Method. Take ten steps starting with $x_0 = 5$.

```
f := x -> x^3 -5*x^2 +3*x -7;
f1 := x -> 3*x^2 - 10*x +3;
x0 := 5.0;
for i from 1 to 100 while ( abs(f(x0)) > 0.000001) do
x0 := x0 - f(x0)/f1(x0);
od;
x0;
f(x0);
```

Output:

$$\begin{aligned} x0 &= 4.678573513 \\ f(x0) &= 0.410 \times 10^{-7} \end{aligned}$$