LECTURE 25

Fast Fourier Transforms

1. Fourier Transforms

Recall that polynomial interpolations (Chapter 6) were introduced as a means of constructing formulae for functions that best replicate given sets of data. Of course, such interpolations only only produce polynomial functions as output, and so might not be well-suited for **all** sets of data. This is particularly true for data that varies very rapidly.

For consider the function

$$f(t) = \sin(2093t)$$

This function function (which by the way represents the oscillations of a piano string tuned to middle C), requires a polynomial of degree at least 4188 to accommodate all the sign flips on the interval between 1 and 2π. Such a high degree polynomial is totally unsuited for subsequent numerical computations because the floating point errors for the leading terms will quickly wash out the lower order terms.

To handle such rapidly varying functions, it makes sense to use as a set of prototypes, a set of functions that have also have the capacity to vary rapidly. The following theorem tells us that trignometric functions can be used as a basis for the expansion of functions that are continuous on the interval $[0, L]$.

THEOREM 25.1. *Let $f(t)$ and $f'(t)$ be continuous on the interval $[0, L]$. Then if we define coefficients*

$$a_n = \frac{2}{L}\int_0^L f(t)\cos(\frac{n\pi t}{L})dt \quad , \quad n = 0, 1, 2, \ldots \tag{25.1}$$

$$b_n = \frac{2}{L}\int_0^L f(t)\sin(\frac{n\pi t}{L})dt \quad , \quad n = 1, 2, 3, \ldots \tag{25.2}$$

we have

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{2n\pi t}{L}\right) + b_n\sin\left(\frac{2n\pi t}{L}\right)\right) \tag{25.3}$$

for all $t \in [0, 2\pi]$

We call refer to the right hand side of (25.3) as the **Fourier expansion of** $f(t)$. We shall use this theorem not so much as a means of computing a given function $f(t)$, but rather as a statement about what the function matching a given set of data might must look like. Just as the goal of polynomial interpolation was to find a suitable set of coefficients A_n so that a data table $\{t_i, f_i\}$ could be replicated by evaluating the polynomial

$$P(t) = \sum_{i=0}^{N} A_i t^i$$

at the node points t_i; our goal here will be to determind a suitable set of Fourier coefficients $\{a_n, b_n\}$ so that the function

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{N}\left(a_n\cos\left(\frac{2n\pi t}{L}\right) + b_n\sin\left(2\frac{n\pi t}{L}\right)\right)$$

replicates a given set of data.

It is both common and convenient to use the Euler formulae

$$\cos(\alpha) = \frac{e^{i\alpha} + e^{-i\alpha}}{2}$$
$$\sin(\alpha) = \frac{e^{i\alpha} - e^{-i\alpha}}{2i}$$

to replace the expansion (25.3) in terms of trignometric functions with an expansion in terms of complex exponential functions

$$f(t) = \sum_{k=-N}^{N} F_k \exp\left(\frac{i2\pi kt}{L}\right)$$

where

$$F_k = \begin{cases} \frac{a_k - ib_k}{2} & , \quad k = 0, 1, 2, 3 \dots \\ \frac{a_k + ib_k}{2} & , \quad k = -1, -2, -3, \dots \end{cases}$$

or alternatively

$$F_k \equiv \frac{2}{L} \int_0^L f(t) e^{i\left(\frac{2k\pi}{L}t\right)} dt$$

In the next section I shall describe a means of computing a suitable collection of coefficients f_k from a given data set.

2. Digital Signal Processing

2.1. Notation. Let $h(t)$ be a functions sampled every τ seconds and let h_n denote the n^{th} sampling: viz.,

$$h_n = h(n\tau) \quad , \quad n = \dots, -3, -2, -1, 0, 1, 2, 3, \dots \tag{25.4}$$

Definition 25.2. *The* sampling rate *is the reciprical of the time interval τ.*

Definition 25.3. *The* Nyquist frequency *f_c is one half the sampling rate:*

$$f_c = \frac{1}{2\tau}$$

Note that the period of a signal at the Nyquist frequency of a sampling is exactly 2τ.

Remark 25.4. Note that two functions $\exp(2\pi i f_1 t)$ and $\exp(2\pi i f_2 t)$ give exactly the same sampling data if and only if the difference between f_1 and f_2 is a multiple of $1/\tau$ which is just the width of the frequency range between $-f_c$ and f_c. For

$$\begin{aligned} \exp(2\pi i f_2 t_n) &= \exp\left(2\pi i \left(f_1 + \frac{m}{\tau}\right)(n\tau)\right) \\ &= \exp(2\pi i (f_1) n\tau + 2mn\pi i) \\ &= \exp(2\pi i (f_1) n\tau) \exp(2nmi\pi) \\ &= \exp(2\pi i (f_1) n\tau) \end{aligned}$$

This phenomenon is called *aliasing*. Accordingly, if sample data at a samplling rate is $\frac{1}{\tau}$ we will not be able to distinguish frequency components f with higher than $f > 2f_c$.

Remark 25.5. For a sampling rate of 8,000 samples per second, $1/8000 = 1.25 \times 10^{-4}$

$$\tau = 1/8000 = 1.25 \times 10^{-4}\text{sec}$$

and the critical frequency is 4000 Hz. (We note that $C7 \approx 4186$ Hz).

2.2. Discrete Fourier Transform. Suppose we sample the values of a function $h(t)$, N consecutive times, starting at time $t = 0$ at intervals of τ seconds:

$$h_k = h(t_k)$$
$$t_k = k\tau \quad , \quad k = 0, 1, 2, 3, \ldots, N-1$$

To make things simplest we shall suppose that N is even. We shall also denote the total time elapsed during the sampling by

$$T = (N-1)\tau \quad .$$

Our goal here is to determine what the frequency spectrum of this function looks like from the sampled data. If $f(t)$ were known at all values of t (not just the discrete set $\{t_k\}$) we would use the Fourier transform

$$H[f] = \int_{-\infty}^{+\infty} h(t) e^{2\pi i f t} dt \tag{25.5}$$

to determine the amplitude of the frequency component f. In the case at hand, we have only N data points, and so we can expect, at most, N independent numbers as output. To extract from (25.5) a reasonable estimate for the amplitude of a component with frequency $f \pm \delta$, we must first make some simplifying assumption about the nature of $h(t)$ so that the integral on the right hand side can be approximated by a finite sum. Thus we assume either

- The support of the function $h(t)$ is completely contained within the interval between $[0, T]$.
- If function $h(t)$ is defined for all t then at least the values of $h(t)$ within the interval $[0, T]$ are typical of what $h(t)$ looks like at other times.

In first case, the first step in reducing the formula to a finite sum is to use one of the assumptions above to replace the improper integral in (25.5) by a integral over a finite length of time

$$H[f] = \int_{-\infty}^{+\infty} h(t) e^{2\pi i f t} dt = \int_{0}^{T} h(t) e^{2\pi i f t} dt$$

We then subdivide the interval $[0, T]$ into $N-1$ subintervals and approximate the integral on the far right by a finite Riemann sum

$$H[f] \approx \sum_{k=0}^{N-1} h(t_k) e^{2\pi i f t_k} \tau = \tau \sum_{k=0}^{N-1} h_k e^{2\pi i f k \tau} \tag{25.6}$$

For the particular frequencies

$$f_n \equiv \frac{n}{N\tau} \quad , \quad n = -\frac{N}{2}, \ldots, \frac{N}{2} \tag{25.7}$$

the sum on the far right reduces to

$$H[f_n] = \tau \sum_{k=0}^{N-1} h_k e^{2\pi i (\frac{n}{N\tau}) k \tau} = \tau \sum_{k=0}^{N-1} h_k e^{2\pi i n k / N}$$

Thus, we obtain

$$H_n \equiv H(f_n) = \tau \sum_{k=0}^{N-1} h_k e^{2\pi i n k / N} \tag{25.8}$$

as a discrete approximation to the Fourier transform and we interpreted the left hand side as the amplitude of the frequency component f_n.

On the other hand, it would seem that if we try to repeat this analysis using instead the second assumption in this same manner that we will be off by an infinite scaling factor because

$$\begin{aligned}\int_{-\infty}^{+\infty} h(t)e^{2\pi ift}dt &= \cdots + \int_{T_i-\tau}^{T_i} h(t)e^{2\pi ift\,dt}dt + \int_{T_i}^{T_f} h(t)e^{2\pi ift\,dt}dt + \int_{T_f}^{T_f+\mathrm{t}} h(t)e^{2\pi ift\,dt}dt + \cdots \\ &\approx \cdots + \int_{T_i}^{T_f} h(t)e^{2\pi ift\,dt}dt + \int_{T_i}^{T_f} h(t)e^{2\pi ift\,dt}dt + \int_{T_i}^{T_f} h(t)e^{2\pi ift\,dt}dt + \cdots\end{aligned}$$

and, indeed, this same formula shows that the original integral can't convery converge either. But actually in applying the second hypothesis, one should really be starting from the formulae

$$h(t) = \sum_{n=0}^{\infty} \frac{1}{N\tau} H_n e^{\frac{2\pi in(t-T_i)}{M\tau}} \tag{25.9}$$

$$H_n = \int_{T_i}^{T_i+N\tau} h(t)e^{\frac{2\pi in(t-T_i)}{N\tau}}dt \tag{25.10}$$

coming from an expansion of $h(t)$ on the interval $(T_i, T_i + N\tau) = (T_i, T_f + \tau)$ in terms of the orthogonal basis functions

$$e_n(t) = e^{\frac{2\pi in(t-T_i)}{N\tau}}$$

(and where the underlying assumption is that $h(T_f + \tau) = h(T_i)$). Ignoring the overall phase factor $e^{2\pi iT_i/N\tau}$ (or to the same effect setting $T_i = 0$) and setting

$$f_n = \frac{n}{N\tau} \tag{25.11}$$

we write

$$\begin{aligned}H_n &= \int_0^{N\tau} h(t)e^{\frac{2\pi if_n t}{N\tau}t}dt \\ &\approx \sum_{k=0}^{N-1} h(t_k)e^{\frac{2\pi if_n t_k}{N\tau}}\tau \\ &= \tau\sum_{k=0}^{N-1} h_k e^{\frac{2\pi ink}{N}}\end{aligned}$$

which is an expression of exactly the same form as (25.8). However, there formulas do not quite say the same thing; in particular, the formula (25.10) is *a priori* valid for $n \in \mathbb{Z}$. However, when we approximate the Riemann integral by a finite Riemann sum, we are effectively making an assumption about the total contribution of high frequency components; in particular, the Nyquist frequency components f_n, with $n = \frac{N}{2}$, can hardly be expected to be nearly constant within time interval τ used of the Riemann sum. The interpretability of the H_n in either case requires an assumption about the attenuation of the high frequency components of the original sampling. Basically, we assume that $|f_n| < f_c = \frac{1}{2\tau}$.

We thus define the discrete Fourier transform of a given set of sample values $\{h_0, h_1, \ldots, h_{N-1}\}$ as the complex number H_n computed via the formula

$$H_n = \sum_{k=0}^{N-1} h_k e^{\frac{2\pi ink}{N}} \tag{25.12}$$

Note that this formula for discrete Fourier transform is periodic in n with period N. Therefore, $H_{-n} = H_{N-n}$. With this conversion in mind, one generally lets the indices n in (25.12) run from 0 to $N-1$, just like the indices of the original sample. In this way the mapping of the N numbers h_k into the N numbers H_n is manifest. However, when this convention is followed you must remember that 0 frequency corresponds to values $n = 0$, that positive frequencies correspond to the values $1 \le n \le \frac{N}{2} - 1$, that the negative frequencies correspond to the values $\frac{N}{2} + 1 \le n \le N - 1$ and that the value $n = \frac{N}{2}$ corresponds to both $f = f_c$ and $f = -f_c$.

3. The Fast Fourier Transform

Note that the discrete Fourier transform (25.12) is a sum of the form

$$H_n = \sum_{k=0}^{N-1} h_k (z^n)^k$$

where

$$z = e^{\frac{2\pi i}{N}}$$