LECTURE 20

Initial Value Problems

We shall now turn our attention to the numerical solution of initial value problems. Recall that an *initial value problem* is a differentiatial equation supplemented by an initial condition:

$$\frac{dx}{dt} = f(x,t) \tag{20.1}$$

$$x(t_0) = x_0 \tag{20.2}$$

Numerical methods for solving initial value problems are extremely important, because analytic solutions of (20.1) exist only in very special cases (linear equations, exact equations, non-exact equations with integrating factors, and equations that are homogeneous of degree 0).

1. Existence and Uniqueness Theorems

Here is the fundamental theorem governing first order initial value problems.

THEOREM 20.1. *Suppose f and $\frac{\partial f}{\partial x}$ are continuous on the rectangle*

$$R = \{(t,x\} \mid |t - t_0| \leq \alpha \ , \ |x - x_o| \leq \beta\}$$

and

$$M = \max_{(t,x)\in R} f(x,t)$$

then the initial value problem

$$\frac{dx}{dt} = f(x,t) \quad , \quad x(t_0) = x_0$$

has a unique solution in the interval

$$\{t \in \mathbb{R} \mid |t - t_o| < \min(\alpha, \beta/M)\}$$

Note that in the solution interval predicted in the theorem above may actually be smaller than the base of the rectangle R. The following theorem allows us to infer the existence and uniqueness of a solution on a prescribed interval $[a,b]$.

THEOREM 20.2. *If $f(x,t)$ is continuous on the strip $a \leq t \leq b$, $-\infty < x < \infty$, and satisfies the inequality*

$$|f(t,x_1) - f(t,x_2)| \leq L\,|x_1 - x_2| \quad , \quad \forall\, x_1, x_2 \in \mathbb{R} \tag{20.3}$$

then

$$\frac{dx}{dt} = f(x,t) \quad , \quad x(t_0) = x_0$$

has a unique solution in the interval $[a,b]$.

Note that the condition (20.3) not only restricts the behavior of $f(t,x_2)$ as $x_2 \to x_1$ (and in fact is stronger than mere continuity at x_1), it also restricts how fast $f(t,x_2)$ can grow as x_2 departs from x_1.

2. Taylor Series Method

Consider an initial value problem of the form

$$\begin{aligned} \frac{dx}{dt} &= f(x,t) \\ x(t_0) &= x_0 \end{aligned}$$

and suppose that the function f on the right hand side has continuous partial derivatives up to at least order n. We then have

$$\begin{aligned} \left.\frac{d^0x}{dt^0}\right|_{t_0} &\equiv x(t_0) = x_0 \\ \left.\frac{d^1x}{dt^1}\right|_{t_0} &\equiv x'(t_0) \\ &= f(t_0, x_0) \\ \left.\frac{d^2x}{dt^2}\right|_{t_0} &\equiv x''(t_0) \\ &= \left.\frac{d}{dt}\left(f(t,x)\right)\right|_{t_0} \\ &= \left.\left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\frac{dx}{dt}\right)\right|_{t_o} \\ &= f_t(t_0, x_0) + f_x(t_0, x_0)\, f(t_0, x_0) \end{aligned}$$

We can continue in this fashion to express all the derivatives $df^{(m)}/dt^{(m)}$ evaluated at t_0 in terms of x_0 and the partial derivatives of f evaluated at (t_0, x_0). We can thus carry out an n^{th} order Taylor expansion of a solution about $x = x_0$

$$x(t) = \sum_{k=0}^{n} \frac{1}{k!} \left.\frac{d^k x}{dt^k}\right|_{t_0} t^k + \mathcal{O}\left(t^{n+1}\right)$$

Example 20.3. Compute the 3^{th} order Taylor expansion of the solution to

$$\begin{aligned} \frac{dx}{dt} &= \cos(t)\sin(x) \\ x(0) &= \frac{\pi}{2}; \end{aligned}$$

We have

$$\begin{aligned} x(0) &= 0 \\ x'(0) &= \cos(0)\sin(x(0)) = 1 \\ x''(0) &= \left.\left(-\sin(t)\sin(x(t)) + \cos(t)\cos(x(t))\,x'(t)\right)\right|_{t=0} \\ &= -(0)(1) + (1)(0)(1) \\ &= 0 \\ x'''(0) &= \left.\left(-\cos(t)\sin(x(t)) - 2\sin(t)\cos(x(t))\,x'(t) - \cos(t)\sin(x(t))\,x'(t) - \cos(t)\cos(x(t))\,x''(t)\right)\right|_{t=0} \\ &= -(1)(1) - 2(1)(0)(1) - (1)(1)(1) - (1)(0)(0) \\ &= -2 \end{aligned}$$

Hence,

$$\begin{aligned} x(t) &= x(0) + x'(0)t + \frac{1}{2}x''(0)t^2 + \frac{1}{6}x'''(0)t^3 + \cdots \\ &= 1 + t - \frac{1}{3}t^3 + \mathcal{O}(t^4) \end{aligned}$$

The preceding algorithm does not lead to such an efficient or accurate numerical method. It is only recently that software that can carry out the necessary symbolic differentiation has appeared. But even if the computer can be taught to carry out the differentiations, may take a large numbers of terms to arrive at a result that accurate for large values of $|t - t_o|$. For error at order n will be

$$err = \frac{C}{(n+1)!}(t - t_o)^n$$

where C is some constant less than or equal to the maximum value of $\left|x^{(n+1)}(t)\right|$ on the interval between t_o and t. In the example, above we would be very hard pressed to expect an accurate answer even on the interval $[0, 1]$.

3. Euler Method

I shall now give an easy method of constructing an (approximate) numerical solution to a differential equation of the form

$$\frac{dx}{dt} = F(t, x) \ , \ \forall\, t \in [a, b] \tag{20.4}$$

The beauty of this method is that it works for any first order differential equation (well, so long as the function $F(x, t)$ on the right hand side is a continuous function of x and t on the interval $[a, b]$). However, it has a rather ugly side as well - the final result will not be a presentation of the solution in terms of known functions; rather it will simply be a table of values of the solution at a discrete set of points $t_i \in [a, b]$.

To construct our numerical solution, we begin by first dividing up the interval $[a, b]$ into n subintervals. Set

$$\Delta t = \frac{b - a}{n} \tag{20.5}$$

and let

$$\begin{aligned} t_0 &= a \\ t_1 &= a + \Delta x \\ t_2 &= a + 2\Delta x \\ &\vdots \\ t_i &= a + i\Delta x \\ &\vdots \\ t_n &= a + n\Delta t = a + \frac{b-a}{\Delta t}\Delta t = b \end{aligned} \tag{20.6}$$

Let $x_i = x(t_i)$ denote the value of a solution of (20.4) at the point t_i and let $\dot{x}_i = \frac{dx}{dt}(t_i)$. The differential equation (20.4) then requires

$$\dot{x}_i = F(t_i, x_i) \ , \ i = 0, 1, \ldots, n \tag{20.7}$$

Now by making Δt small enough, we can approximate $\dot{x}_i = \frac{dx}{dt}(t_i)$ to an arbitrarily high degree of accuracy by setting

$$\dot{x}_i = \frac{dx}{dt}(t_i) \approx \frac{\Delta x}{\Delta t} = \frac{x_{i+1} - x_i}{\Delta t} \tag{20.8}$$

And so, the differential equation (20.4) is approximately equivalent to the following set of algebraic equations

$$\frac{x_{i+1} - x_i}{\Delta t} = F(t_i, x_i) \ , \ i = 0, \dots, n-1 \tag{20.9}$$

Solving (20.9) for x_{i+1}, we obtain

$$x_{i+1} = x_i + \Delta t F(t_i, x_i) \ . \ i = 0, 1, \dots, n-1 \tag{20.10}$$

or, more explictly,

$$\begin{aligned}
x_1 &= x_0 + \Delta t F(t_0, x_0) && (20.11)\\
x_2 &= x_1 + \Delta t F(t_1, x_1) && (20.12)\\
x_3 &= x_2 + \Delta t F(t_2, x_2) && (20.13)\\
&\vdots && (20.14)\\
x_{i+1} &= x_i + \Delta t F(t_i, x_i) && (20.15)\\
&\vdots && (20.16)\\
x_n &= x_{n-1} + \Delta t F(t_{n-1}, x_{n-1}) && (20.17)
\end{aligned}$$

This set of equations relates now relates all the x_i, $i = 1, 2, \dots, n$ to x_0.

To see this, note that when $i = 0$ equation (20.11) implies

$$x_1 = x_0 + F(t_0, x_0) \tag{20.18}$$

But now inserting this expression for x_1 into the right hand side of (20.12) yields

$$x_2 = x_0 + F(t_0, x_0) + F(t_1, x_0 + F(t_0, x_0)) \tag{20.19}$$

Thus, x_2 is expressed entirely in terms of x_0. We now replace the x_2 on the rightt hand side of (20.13) with the expression on the right hand side of (20.19) to express x_3 directly in terms of x_0. Repeating this process $n-1$ times we can express all the x_i in terms of x_0.

Example 20.4. Construct a numerical solution of the differential equation

$$\frac{dx}{dt} = x^2 t \ , \ \forall \ t \in [0.1].$$

such that

$$x(0) = 1.$$

on the interval $[0, 1]$.

Let's set $n = 10$, and let

$$\begin{aligned}
\Delta t &= \frac{1-0}{n} = .\frac{1}{10}\\
\\
t_0 &= 0\\
t_1 &= t_0 + \Delta t = 0.1\\
t_2 &= t_0 + 2\Delta t = 0.2\\
\\
&\vdots\\
t_{10} &= t_0 + 10\Delta t = 1
\end{aligned}$$

and let x_i, $i = 0, \dots, 10$ represent the values of $x(t)$ when $t = 0, \dots, 10$. Since in this example

$$F(t, x) = x^2 t$$

equations (20.11) - (20.17) take the form

$$
\begin{aligned}
x_1 &= x_0 + t_0 x_0^2 \Delta t \\
x_2 &= x_1 + t_1 x_1^2 \Delta t \\
x_3 &= x_2 + t_2 x_2^2 \Delta t \\
&\vdots \\
x_{10} &= x_9 + t_9 x_9^2 \Delta t
\end{aligned}
$$

Since $\Delta t = \frac{1}{10}$, $t_i = \frac{i}{10}$ and $x_0 = x(0) = 1$, in this example, these equations can also be written as

$$
\begin{aligned}
x_1 &= 1 \\
x_2 &= x_1 + \frac{0.1}{10} x_1^2 = 1 + (0.01)(1) = 1.01 \\
x_3 &= x_2 + \frac{0.2}{10} x_2^2 = (1.01) + (0.02)(1.01) = 1.0302 \\
&\vdots \\
x_{10} &= x_9 + \frac{0.9}{10} x_9^2 = 1.712852586
\end{aligned}
$$

Alternatively, we can choose our number of sample points n to very large, say $n = 1000$, repeat the calculation (on a computer) and plot the results. Doing so we get a graph like

which is not only far more accurate (in matching the exact solution), but also contains so many data points that we don't even have to imagine connecting them to see the graph of $x(t)$.

Below I give a simple Maple routine that automated this calculation:

```
n:= 100:
t[0] := 0.0:
x[0] := 1.0:
f := (x,t) -> t*x^2:
dt := (1.0)/n:
```

```
for i from 1 to n do
      t[i] := i*dt:
      x[i] := x[i-1] + dt*f(x[i-1],t[i-1]):
od:
datapoints := {seq([t[i],x[i]],i=0..n)}:
with(plots):
pointplot(datapoints);
```