LECTURE 15

Interpolating Functions

In the next series of lectures we will discuss methods for finding functions that best fit a given set of data. We shall call such functions *interpolating functions*, and we shall consider several different methods for finding such functions. We shall begin this discussion with the problem of interpolating data by means of polynomial functions.

1. Polynomial Interpolation

Consider the following problem:

PROBLEM 15.1. *Given a table of $n+1$ distinct data points (x_i, y_i), $i = 0, \ldots, n$, find a polynomial of P of lowest degree for which*

$$P(x_i) = y_i \quad \forall\, i \tag{15.1}$$

That this problem **has** a solution is fairly easy to see. For if we set

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

then, so long as the x_i are all distinct, equations (15.1) will constitute a system of $n+1$ linearly independent equations

$$\begin{aligned} (x_0)^n a_n + (x_0)^{n-1} a_{n-1} + \cdots + x_0 a_1 + a_0 &= y_0 \\ (x_1)^n a_n + (x_1)^{n-1} a_{n-1} + \cdots + x_1 a_1 + a_0 &= y_1 \\ &\vdots \\ (x_n)^n a_n + (x_n)^{n-1} a_{n-1} + \cdots + x_n a_1 + a_0 &= y_n \end{aligned} \tag{15.2}$$

for $n+1$ unknowns $a_0, \ldots, a_n$. (That these equations are linearly independent may not be immediately obvious; but it follows from the fact that the monomials x^i, $i = 0, \ldots, n$ are linearly independent functions.) We can thus expect a unique solution of degree n.

The formal linear algebraic argument for the existence of solutions is, however, not really the best way to finding a solution. For, it takes on the order of n^3 operations to solve a linear system such as (15.1) on a computer, if we had a million or so data points to fit, we would need to carry out at least 10^{18} operations to calculate a solution. While such a calculation might actually be feasible on modern hardware, there are much better ways to procede.

2. The Newtonian Form of the Interpolation Polynomial

The following algorithm allows one to build up an interpolating polynomial much more quickly. Let

$$P_0(x) = y_0$$

This function clearly satisfies $P_0(x_0) = y_0$, but it won't satisfy $P_0(x_i) = y_i$ unless $y_i = y_0$. The next step is to write down a function $P_1(x)$ that satisfies at least $P_1(x_0) = y_0$ and $P_1(x_1) = y_1$. This can be had by simply adding a term to P_0

$$P_1(x) = y_0 + \ c_1(x - x_0)$$

and choosing c so that

$$P_1(x_1) = y_1 \quad \Rightarrow \quad c_1 = \frac{y_1 - y_0}{x_1 - x_0}$$

Now we have a polynomial function that at least agrees with the data at two points. To create a function that is correct for the first three data points we set

$$P_2(x) = P_1(x) + c_2\,(x - x_0)\,(x - x_1)$$

and choose c so that

$$y_2 = P_2(x_2) \quad \Rightarrow \quad c_2 = \frac{y_2 - P_1(x_2)}{(x_2 - x_0)\,(x_2 - x_1)}$$

Note that the reason why this procedure words is that we have a function $P_1(x)$ that is already correct at the first two points and we are adding to it a function that makes no contribution to the values at x_0 and x_1 but which can be adjusted to correct value y_2 at x_2.

Now suppose we have carried out this procedure to construct a (degree k) polynomial $P_k(x)$ that replicates the first k data points. To obtain a polynomial function that replicates all the data points up to (x_k, y_k) we set

$$P_k(x) = P_{k-1}(x) + c_k\,(x - x_0)\,(x - x_1)\cdots(x - x_{k-1})$$

where c_k is chosen so that $P_k\,(x_k) = y_k$

$$\Rightarrow \quad c_k = \frac{y_k - P_{k-1}(x_k)}{(x_k - x_0)\,(x_k - x_1)\cdots(x_k - x_{k-1})}$$

Clearly, so long as the points x_i, $i = 0, \ldots, n$ are all distinct, there is no obstruction to this program and so we'll be able to construct a polynomial $P_n(x)$ of degress $m \leq n$ that interpolates the data (it could happen that some of the numbers $c_i = 0$, that's why perhaps the degree of $P_n(x)$ might be less than n).

EXAMPLE 15.1. Find the Newton form of the interpolation polynomial for the following set of data

$$\begin{array}{ll} x_0 = 0 & y_0 = -1 \\ x_1 = 1 & y_1 = -1 \\ x_2 = 2 & y_2 = 1 \\ x_3 = 3 & y_3 = 11 \end{array}$$

- Set

  $$P_0(x) = y_0 = -1$$

  Then

  $$\begin{aligned} c_1 &= \frac{y_1 - P_0(x)}{(x_1 - x_0)} = \frac{-1 - (-1)}{1 - 0} = 0 \\ &\Rightarrow \quad P_1(x) = P_0(x) + c_1(x - x_0) = -1 + 0 = -1 \end{aligned}$$

  $$\begin{aligned} c_2 &= \frac{y_2 - P_1(x)}{(x_2 - x_1)(x_2 - x_0)} = \frac{1 - (-1)}{(2 - 1)(2 - 0)} = 1 \\ &\Rightarrow \quad P_2(x) = P_1(x) + c_2(x - x_0)(x - x_1) = -1 + (x - 1)x \end{aligned}$$

  $$\begin{aligned} c_3 &= \frac{y_3 - P_2(x)}{(x_3 - x_2)(x_3 - x_1)(x_3 - x_0)} = \frac{11 - 5}{(3 - 2)(3 - 1)(3 - 0)} = 1 \\ &\Rightarrow \quad P_3(x) = P_2(x) + c_3(x - x_0)(x - x_1)(x - x_2) = -1 + (x - 1)x + (x - 2)(x - 1)x \end{aligned}$$

So the Newton form of the interpolation polynomial is

$$P(x) = P_3(x) = -1 + (x-1)x + (x-2)(x-1)x$$

Noting that

$$-1 + (x-1)x + (x-2)(x-1)x = x^3 - 2x^2 + x - 1$$

one can easily verify that $P(x)$ correctly interpolates the given data. □

It should be also fairly obvious as to how one might write a program that would calculate, in an iterative fashion, all the coefficients c_k, all the intermediary polynomials $P_k(x)$, and finally the polynomial

$$\begin{aligned} P_n(x) &= P_{n-1}(x) + c_n(x-x_0)(x-x_1)\cdots(x-x_{n-1}) \\ &= c_0 + c_1(x-x_0) + c_2(x-x_0)(x-x_1) + \cdots + c_n(x-x_0)(x-x_1)\cdots(x-x_{n-1}) \end{aligned}$$

that interpolates all the data points. However, we shall not do so; because it turns out that *this* algorithm, though useful in seeing the why and how of polynomial interpolation, is not all that efficient in producing the final interpolating polynomial. Instead, we shall view the algorithm above as a constructive proof of the following theorem.

THEOREM 15.2. *Let $x_0, x_1, \ldots, x_n$ be distinct real numbers and let $y_0, y_1, \ldots, y_n$ be a corresponding (not necessarily distinct) set of values. Then there is a unique polynomial $P_n(x)$ of degree at most n such that*

$$P_n(x_i) = y_i \quad , \quad i = 0, 1, \ldots, n$$

REMARK 15.3. Actually, we have not yet proved the uniqueness of $P_n(x)$. As this will be important latter on, I'll give the argument here. Suppose $Q(x)$ and $P(x)$ were two polynomials of degreee at most n that interpolated the same set of data points $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$. Then the polynomial $P(x) - Q(x)$ would be of degree less than or equal to n and would have the property that

$$P(x_i) - Q(x_i) = y_i - y_i = 0 \quad , \quad i = 0, 1, \ldots, n$$

But a non-zero polynomial of degree n can have at most n distinct zeros. Therefore, we must have $P(x) - Q(x) = 0$, hence $P(x) = Q(x)$.

Henceforth, we shall refer to the presentation

$$\begin{aligned} P(x) &= c_0 + c_1(x-x_0) + c_2(x-x_0)(x-x_1) + \cdots + c_n(x-x_0)(x-x_1)\cdots(x-x_{n-1}) \\ &= \sum_{i=0}^{n} c_i \left(\prod_{j=0}^{i-1} (x-x_j) \right) \end{aligned}$$

of the interpolating polynomial as the **Newton form of the interpolation polynomial**.

3. The Lagrange Form of the Interpolation Polynomial

We shall now present another way writing down the polynomial that interpolates a given set of data. The basis for this presentation of the interpolation polynomial lies in the fact that the final coefficients of each power of x in $P(x)$ depend linearly on the data points y_i. To see this note that the equations (15.2) for the coefficients a_i can be written

$$\begin{pmatrix} (x_0)^n & (x_0)^{n-1} & \cdots & 1 \\ (x_1)^n & (x_1)^{n-1} & \cdots & 1 \\ \vdots & & & \\ (x_n)^n & (x_n)^{n-1} & & 1 \end{pmatrix} \begin{pmatrix} a_n \\ a_{n-1} \\ \vdots \\ a_0 \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{pmatrix}$$

If we think of this as a matrix equation of the form $\mathbf{Xa} = \mathbf{y}$, then its solution can be represented formally as $\mathbf{a} = \mathbf{X}^{-1}\mathbf{y}$ and so each coefficient a_i would depend linearly on the values y_j, $j = 0, \ldots, n$. We can then collect all the terms in the interpolation polynomial that are proportional to each of the y_i to obtain an expression of the form

$$P(x) = y_0\ell_0(x) + y_1\ell_1(x) + \cdots + y_n\ell_n(x)$$

Now the polynomials $\ell_i(x)$ depend only on the variables data points x_i and not at all on their values y_i. Therefore, by looking at special sets of potential data, we can figure out what they must be. For example, suppose we fix i and demand

$$y_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \tag{15.3}$$

Then the interpolating polynomial for this set of data would look like

$$P(x) = \sum_{j=0}^{n} y_j\ell_j(x) = l_i(x)$$

and so we could conclude that, since this polynomial interpolates the data (15.3)

$$0 = P(x_j) = \ell_i\,(x_j) \quad \text{for all } i \neq j$$

which tells us that $\ell_i(x)$ has n distinct roots x_j, $i \in \{0, 1, \ldots, n \mid i \neq j\}$. Since the degree of $\ell_i(x)$ is at most n, and because the trivial solution $\ell_i(x)$ is not allowed (otherwise $P(x_i) \neq 1$), and because the interpolating polynomial must be unique we can conclude that

$$l_i(x) = \prod_{j \neq i} \frac{(x - x_j)}{(x_i - x_j)}$$

simply because it has the property that it interpolates the data (15.3).

Having identified the **cardinal functions** $\ell_i(x)$, we can now write

$$P(x) = \sum_{i=0}^{n} y_i\ell_i(x) = \sum_{i=0}^{n} y_i \left(\prod_{j \neq i} \frac{(x - x_j)}{(x_i - x_j)} \right)$$

This presentation of the interpolation polynomial is known as the **Lagrange form of the interpolation polynomial**.

EXAMPLE 15.4. Find the Lagrange form of the interpolation polynomial for the following set of data

$$\begin{array}{ll} x_0 = 0 & y_0 = -1 \\ x_1 = 1 & y_1 = -1 \\ x_2 = 2 & y_2 = 1 \\ x_3 = 3 & y_3 = 11 \end{array}$$

- Writing down the Lagrange form of the interpolation is pretty straight-forward.

$$\begin{aligned}
P(x) &= \sum_{i=0}^{n} y_i \left(\prod_{j \neq i}^{n} \frac{(x - x_j)}{(x_i - x_j)} \right) \\
&= y_0 \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} \\
&\quad + y_1 \frac{(x - x_0)(x - x_2)(x - x_3)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)} \\
&\quad + y_2 \frac{(x - x_0)(x - x_1)(x - x_3)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)} \\
&\quad + y_3 \frac{(x - x_0)(x - x_1)(x - x_2)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)} \\
&= -\frac{(x-1)(x-2)(x-3)}{(-1)(-2)(-3)} - \frac{x(x-2)(x-3)}{(1)(-1)(-2)} \\
&\quad + \frac{x(x-1)(x-3)}{(2)(1)(-1)} + (11)\frac{x(x-1)(x-2)}{(3)(2)(1)}
\end{aligned}$$

The final expression is the Lagrange form of the interpolation polynomial. While tedious, it is nevertheless straightforward to verify that

$$\begin{aligned}
P(x) &= -\frac{(x-1)(x-2)(x-3)}{(-1)(-2)(-3)} - \frac{x(x-2)(x-3)}{(1)(-1)(-2)} + \frac{x(x-1)(x-3)}{(2)(1)(-1)} + (11)\frac{x(x-1)(x-2)}{(3)(2)(1)} \\
&= x^3 - 2x^2 + x = 1
\end{aligned}$$

and so we recover the same interpolation polynomial as in the first example. $\square$