LECTURE 14

Solutions to First Exam

1. Suppose $\{e_n\}$ is a positive sequence converging to zero such that

$$e_{n+1} = e_n e_{n-1}$$

Show that the rate of convergence of $\{e_n\}$ is of order $\alpha \approx 1.6$. (Hint, assume a relationship of the form $e_{n+1} = C\,(e_n)^\alpha$ and determine α.)

- Set

$$\begin{aligned} e_{n+1} &= C(e_n)^\alpha \quad \Rightarrow \quad e_n = C\,(e_{n-1})^\alpha \\ &\Rightarrow \quad e_{n-1} = \left(\frac{e_n}{C}\right)^{1/\alpha} = C^{-1/\alpha}\,(e_n)^{1/\alpha} \end{aligned}$$

Plugging these expressions for e_{n+1} and e_{n-1} into the relation $e_{n+1} = e_n e_{n-1}$ yields

$$C\,(e_n)^\alpha = C^{-1/\alpha}\,(e_n)^{1/\alpha}\,e_n$$

or

$$C^{\alpha+1/\alpha} = (e_n)^{1+\frac{1}{\alpha}-\alpha}$$

Since the right hand side must remain constant as $n \to \infty$ we require

$$0 = 1 + \frac{1}{\alpha} - \alpha \quad \Rightarrow \quad \alpha^2 - \alpha - 1 = 0 \quad \Rightarrow \quad \alpha = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1}{2} \pm \frac{\sqrt{5}}{2} = 1.62\ldots, -0.618\ldots$$

Since the series will not converge if α is negative, we must have $\alpha = 1.62\ldots$

$\square$

2. Suppose on a given computer floating point numbers are stored as 16-bit data types in the form

$$x = (-1)^s \times (1.m)_2 \times 2^e$$

where s is a 1 bit string for prescribing the sign s of a number, m is a 7 bit string for prescribing the (normalized) mantissa of the corresponding floating point number, and 8 bits reserved for prescibing the exponent e. Given that $x = \frac{12}{10} = (1.2)_{10} = \left(1.001100110\overline{011}\right)_2$ determine the (normalized) mantissa $1.m$ of the machine number closest to x.

- The machine number immediately below x is obtained by truncating the binary decimal expansion of x to 7 significant figures past the leading 1:

$$q_- = (1.0011001)_2 = 1 + 2^{-3} + 2^{-4} + 2^{-7} = 1 + 0.125 + 0.0625 + 0.0078125 = 1.1953125$$

The mantissa of the machine number immediately above x is obtained by increasing the last digit of q_- by 1:

$$q_+ = q_- + 2^{-7} = 1.1953125 + 0.0078125 = 1.203125$$

We have

$$\begin{aligned} |x - q_-| &= 0.0046875 \\ |x - q_+| &= 0.0031250 \end{aligned}$$

so q_+ is closer to x than q_-. Hence the machine number corresponding to x is $q_+ = 1.203125$

$\square$

3. Starting with $a = 1$, $b = 2$, carry out 3 iterations of the bisection method to find an approximate root of $x^3 = 2$.

- We have
$$f(x) = x^3 - 2$$
and
$$\begin{aligned}
a_0 &= 1 \quad , \quad b_0 = 2 \quad , \quad c_0 = \frac{1+2}{2} = 1.5 \\
&\Rightarrow \quad f(c_0) = 1.375 > 0 \quad \Rightarrow \quad a_1 = 1 \quad , \quad b_1 = 1.5 \\
a_1 &= 1 \quad , \quad b_1 = 1.5 \quad , \quad c_1 = \frac{1+2}{2} = 1.25 \\
&\Rightarrow \quad f(c_0) = -0.046875 < 0 \quad \Rightarrow \quad a_2 = 1.25 \quad , \quad b_2 = 1.5 \\
a_2 &= 1.25 \quad , \quad b_2 = 1.5 \quad , \quad c_0 = \frac{1+2}{2} = 1.375 \\
&\Rightarrow \quad f(c_0) => 0.599609 \quad \Rightarrow \quad a_3 = 1.25 \quad , \quad b_1 = 1.375 \\
x &\approx c_3 = 1.3125
\end{aligned}$$

$\square$

4. Starting with $x_0 = 2$ carry out 2 iterations of Newton's method to find a solution of $x - \sin(x) = 1$.

- We have
$$\begin{aligned}
f(x) &= x - \sin(x) - 1 \\
f'(x) &= 1 - \cos(x)
\end{aligned}$$
So
$$\begin{aligned}
x_1 &= x_0 + \frac{f(x_0)}{f'(x_0)} = 2 - \frac{0.090702573}{1.416146837} = 1.935951152 \\
x_2 &= x_1 + \frac{f(x_1)}{f'(x_1)} = 1.935951152 - \frac{0.001882667}{1.357093916} = 1.934563874
\end{aligned}$$

$\square$

5. Solve the following linear system

$$\mathbf{Ax} = \begin{pmatrix} 2 & -1 & 1 \\ 0 & 1 & -1 \\ 2 & -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

using the fact that $\mathbf{A} = \mathbf{LU}$ with

$$\mathbf{L} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \quad , \quad \mathbf{U} = \begin{pmatrix} 2 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

- We first solve

  $$\mathbf{Lz} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \mathbf{b}$$

  for $\mathbf{z}$. This leads us to

  $$\left.\begin{array}{ll} z_1 & = 1 \\ z_2 & = 0 \\ z_1 + z_3 & = 1 \end{array}\right\} \quad \Rightarrow \quad \mathbf{z} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

  We now solve $\mathbf{Ux} = \mathbf{z}$ or

  $$\begin{pmatrix} 2 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

  or

  $$\left.\begin{array}{ll} 2x_1 - x_2 + x_3 & = 1 \\ x_2 - x_3 & = 0 \\ x_3 & = 0 \end{array}\right\} \quad \Rightarrow \quad \mathbf{x} = \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \end{pmatrix}$$

$\square$

6. Use Gaussian elimination to determine an LU factorization of

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & 3 \\ 1 & 3 & 6 \end{pmatrix}$$

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$$\begin{aligned} & \begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & 3 \\ 1 & 3 & 6 \end{pmatrix} \\ \Rightarrow \quad & \begin{pmatrix} 1 & 1 & 1 \\ \underline{1} & 2 & 2 \\ \underline{1} & 2 & 5 \end{pmatrix} \\ \Rightarrow \quad & \begin{pmatrix} 1 & 1 & 1 \\ \underline{1} & 2 & 2 \\ \underline{1} & \underline{1} & 3 \end{pmatrix} \end{aligned}$$

$$\Rightarrow \quad \mathbf{L} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \quad , \quad \mathbf{U} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}$$

$\square$