LECTURE 1

Variations on Taylor's Formula

Numerical methods are *ipso facto* approximate methods. This being the case, it will be important throughout this course to determine the accuracy of numerical results. We shall begin by reviewing the analytic methods by which we approximate functions and how we bound the errors that arise from such approximations.

DEFINITION 1.1. *We denote by $C^n[a,b]$ the set of functions on the interval $[a,b] \subset \mathbb{R}$ that have continuous derivatives up to order n. We denote by $C^n(\mathbb{R})$ the set of functions on the real line that have continuous derivatives up to order n. We denote by $C^\infty[a,b]$ and $C^\infty(\mathbb{R})$ the sets of functions for which derivatives of all orders exist on, respectively, $[a,b]$ and $\mathbb{R}$.*

EXAMPLE 1.2. If $f(x) \equiv x^2 \sin(1/x)$ then f is in $C^1(\mathbb{R})$ but not in $C^2(\mathbb{R})$. To see this, note

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}\left(x^2 \sin\left(\frac{1}{x}\right)\right) = 0$$

since $x^2 \to 0$ and $\sin\left(\frac{1}{x}\right)$ is bounded between -1 and 1. Thus $f \in C^0(\mathbb{R})$.

$$\begin{aligned} \left.\frac{df}{dx}\right|_{x=0} &\equiv \lim_{x\to 0} \frac{f(x) - f(0)}{x} \\ &= \lim_{x\to 0}\left(x \sin\left(\frac{1}{x}\right)\right) \\ &= 0 \end{aligned}$$

again because it the limit of a function that is the product of a function that vanishes at the limit point and a function that remains bounded. So f is also in $C^1(\mathbb{R})$. However,

$$\begin{aligned} \left.\frac{d^2f}{dx^2}\right|_{x=0} &\equiv \lim_{x\to 0} \frac{f'(x) - f'(0)}{x} \\ &= \lim_{x\to 0} \frac{2x\sin(1/x) - \cos(\frac{1}{x}) - 0}{x} \\ &= \lim_{x\to 0} \frac{-\cos(\frac{1}{x})}{x} \end{aligned}$$

does not exist. Thus, $f \notin C^2(\mathbb{R})$. □

REMARK 1.3. In general, we have

$$C^\infty[a,b] \subset \cdots \subset C^3[a,b] \subset C^2[a,b] \subset C^1[a,b] \subset C^0[a,b]$$

THEOREM 1.4. ***(Taylor's Theorem with Integral Remainder).*** *If $f \in C^{n+1}[a,b]$, then for any points $x, x_o \in [a,b]$,*

$$f(x) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(x_o)(x - x_o)^k + R_n(x)$$

where

$$R_n(x) \equiv \frac{1}{n!}\int_{x_o}^{x} f^{(n+1)}(t)(x-t)^n \, dt$$

Proof. This theorem is surprisingly easy to prove. We start by using integration by parts to evaluate the right hand side of the definition of the error term $R_n(x)$. Setting

$$\begin{array}{rr} u = \frac{1}{n!}(x-t)^n & dv = f^{(n+1)}(t)dt \\ du = -\frac{1}{(n-1)!}(x-t)^{n-1} & v = f^{(n)}(t) \end{array}$$

and using the integration by parts formula

$$\int_{x_o}^{x} udv = uv|_{x_o}^{x} - \int_{x_o}^{x} vdu$$

we find

$$\begin{aligned} R_n(x) &\equiv \frac{1}{n!}(x-t)^n f^{(n)}(t)\Big|_{x_o}^{x} + \frac{1}{(n-1)!}\int_{x_o}^{x} f^{(n)}(t)(x-t)^{n-1}dt \\ &= -\frac{1}{n!}f^{(n)}(x_o)(x-x_o)^n + R_{n-1}(x) \end{aligned}$$

We thus have a recursive formula for $R_n(x)$. Using this recursive formula over and over again we can reduce the right hand side to

$$\begin{aligned} R_n(x) &= -\frac{1}{n!}f^{(n)}(x_o)(x-x_o)^n - \frac{1}{(n-1)!}f^{(n-1)}(x_o)(x-x_o)^{n-1} + \cdots \\ &\quad \cdots - \frac{1}{1!}f'(x_o)(x-x_o)^n + \frac{1}{0!}R_0(x) \\ &= -\sum_{n=1}^{n} f^{(n)}(x_o)(x-x_o)^n + \frac{1}{0!}\int_{x_o}^{x} f'(t)(x-t)^0 dt \\ &= -\sum_{n=1}^{n} f^{(n)}(x_o)(x-x_o)^n + \int_{x_o}^{x} f'(t)dt \\ &= -\sum_{n=1}^{n} f^{(n)}(x_o)(x-x_o)^n + f(x) - f(x_o) \\ &= -\sum_{n=0}^{n} f^{(n)}(x_o)(x-x_o)^n + f(x) \end{aligned}$$

Solving the extreme sides for $f(x)$ yields

$$f(x) = \sum_{k=0}^{n} \frac{1}{k!}f^{(k)}(x_o)(x-x_o)^k + R_n(x)$$

□

REMARK 1.5. Note that in this formulation of Taylor's Theorem, we have an explicit formula for computing the error term $R_n(x)$. However, since the error term depends on the choice of x and x_o in a non-trivial way, we do not have (at least at face value) an understanding of how the error term changes as we vary x. Such an understanding would be critical if we are to regard

$$f_n(x) \equiv \sum_{k=0}^{n} \frac{1}{k!}f^{(k)}(x_o)(x-x_o)^k$$

as a polynomial function *approximate* to the original function $f(x)$. Below we'll deduce bounds on the error term $R_n(x)$ as x ranges throughout the interval $[a,b]$.

THEOREM 1.6. ***(Mean Value Theorem for Integrals)****. Let u and v be continuous real-valued functions on an interval $[a,b]$, and suppose that $v(x) \geq 0$ for all $x \in [a,b]$. Then there exists a point $\xi \in [a,b]$ such that*

$$\int_{a}^{b} u(x)v(x)dx = u(\xi)\int_{a}^{b} v(x)dx \quad .$$

Proof. Let α and β denote, respectively, the minimal and maximal values of $u(x)$ on $[a,b]$. Then

$$\alpha v(x) \leq u(x)v(x) \leq \beta v(x) \quad , \quad \forall\, x \in [a,b]$$

Integrating this relationship between a and b yields

$$\alpha \int_a^b v(x)dx \leq \int_a^b u(x)v(x)dx \leq \beta \int_a^b v(x)dx$$

or

$$\alpha \leq \frac{\int_a^b u(x)v(x)dx}{\int_a^b v(x)dx} \leq \beta$$

Now by hypothesis, $\alpha = u(x_1)$ and $\beta = u(x_2)$ for some $x_1, x_2 \in [a,b]$. By the Intermediate Value Theorem for Continuous Functions, for any number γ between α and β there must exist a point $\xi \in [a,b]$ such that $u(\xi) = \gamma$. In particular, there must exist a point $\xi \in [a,b]$ such that

$$u(\xi) = \frac{\int_a^b u(x)v(x)dx}{\int_a^b v(x)dx}$$

Hence there exists a point $\xi \in [a,b]$ such that

$$\int_a^b u(x)v(x)dx = u(\xi) \int_a^b v(x)dx$$

□

REMARK 1.7. The usual Mean Value Theorem; i.e., the statement that if $f(x)$ is continuous and differentiable on $[a,b]$ then there is a point $\xi \in [a,b]$ such that

$$f'(\xi) = \frac{f(b) - f(a)}{b - a}$$

is a special case of the formulation above. To see this, note that if we take $u(x) = f'(x)$ and $v(x) = 1$, then the above theorem implies

$$\int_a^b f'(x)dx = f'(\xi) \int_a^b dx$$

Carrying out the integrations on both sides yields

$$f(b) - f(a) = f'(\xi)\,(b - a) \quad \Rightarrow \quad f'(\xi) = \frac{f(b) - f(a)}{b - a}$$

THEOREM 1.8. ***(Taylor's Theorem with Lagrange Remainder)****. If $f \in C^n[a,b]$ and $f^{n+1}(x)$ exists on (a,b) then for any point x and x_o in $[a,b]$*

$$f(x) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(x_o)(x - x_o)^k + E_n(x)$$

where

$$E_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi)\,(x - x_o)^{n+1}$$

for some point ξ between x and x_o.

Proof. According to Taylor's Theorem with Integral Remainder is

$$f(x) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(x_o)(x - x_o)^k + \frac{1}{n!} \int_{x_o}^{x} f^{(n+1)}(t)(x - t)^n\, dt$$

By the Mean Value Theorem for Integrals

$$\frac{1}{n!} \int_{x_o}^{x} f^{(n+1)}(t)(x - t)^n\, dt = \frac{1}{n!} f^{(n+1)}(\xi) \int_{x_o}^{x} (x - t)^n\, dt = \frac{1}{(n+1)!} f^{(n+1)}(\xi)\,(x - x_o)^{n+1}$$

for some ξ between x and x_o. Hence,

$$f(x) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(x_o)(x - x_o)^k + \frac{1}{(n+1)!} f^{(n+1)}(\xi)\,(x - x_o)^{n+1}$$

for some ξ between x and x_o. □

Remark 1.9. Note that in this formulation of Taylor's theorem the error term is still not precisely determined; because the point ξ in the theorem statement is left undetermined. We know only that there is some point $\xi \in [a, b]$ such that

$$f(x) - \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(x_o)(x - x_o)^k = \frac{1}{(n+1)!} f^{(n+1)}(\xi)\,(x - x_o)^{n+1}$$

However, what we gain in this formulation are solid bounds on the size of the error term over a range of x. To see this let M be the maximum value of $\left|f^{(n+1)}(x)\right|$ on the interval $[a, b]$, then

$$\left| f(x) - \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(x_o)(x - x_o)^k \right| \leq \frac{M}{(n+1)!} \left|x - x_o\right|^{n+1}$$

Note that this bound is independent of the choice of x and x_o in $[a, b]$.

Corollary 1.10. ***(Alternative Form of Taylor's Theorem.)*** *If $f \in C^{n+1}[a, b]$, then for any points x and $x + h \in [a, b]$,*

$$f(x + h) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(x) h^k + E_n(h)$$

where

$$E_n(h) = \frac{1}{(n+1)!} f^{(n+1)}(\xi) h^{n+1}$$

in which the point ξ lies between x and $x + h$.

1. Problems

1.1. Given that

$$\frac{d^n}{dx^n} \left(\ln |x|\right)\Big|_{x=1} = (-1)^{n-1}(n-1)!$$

(a) Use the Taylor Theorem with Integral Remainder to find the magnitude of the error term $R_{100}(1.99)$ when one approximates $\ln[1.99]$ using the first 101 terms of the Taylor expansion about 1 of $\ln|x|$.

(b) Use the Taylor Theorem with Lagrange Remainder to obtain an upper bound on the error term $E_{100}(x)$ when x ranges from 1.985 to 1.995 for the Taylor expansion of $\ln|x|$ about 1.