Lecture 15

Laplace’s Equation on a Disc

Last time we solved the Dirichlet problem for Laplace’s equation on a rectangular region. Today we’ll look at the corresponding Dirichlet problem for a disc.

Thus, we consider a disc of radius $a$

\begin{equation}
D = \{ [x, y] \in \mathbb{R}^2 \mid x^2 + y^2 = a^2 \}
\end{equation}

upon which the following Dirichlet problem is posed:

\begin{align}
(2a) \quad u_{xx} + u_{yy} &= 0, \quad \forall \ [x, y] \in D \\
(2b) \quad u(a \cos \theta, a \sin \theta) &= h(\theta), \quad 0 \leq \theta \leq 2\pi
\end{align}

We shall solve this problem by first rewriting Laplace’s equation in terms of polar coordinates (which are most natural to the region $D$) and then separating variables and proceeding as in Lecture 14.

Now under the change of variables

\[
x = r \cos \theta \\
y = r \sin \theta
\]

we have

\[
\frac{\partial}{\partial x} = \frac{\partial r}{\partial x} \frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x} \frac{\partial}{\partial \theta} = \frac{x}{\sqrt{x^2 + y^2}} \frac{\partial}{\partial r} - \frac{y}{x^2 + y^2} \frac{\partial}{\partial \theta} = \cos \theta \frac{\partial}{\partial r} - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta}
\]

\[
\frac{\partial}{\partial y} = \frac{\partial r}{\partial y} \frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y} \frac{\partial}{\partial \theta} = \frac{y}{\sqrt{x^2 + y^2}} \frac{\partial}{\partial r} + \frac{x}{x^2 + y^2} \frac{\partial}{\partial \theta} = \sin \theta \frac{\partial}{\partial r} + \frac{\cos \theta}{r} \frac{\partial}{\partial \theta}
\]

After a short but tedious calculation one finds

\begin{equation}
\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}
\end{equation}

and so in terms of polar coordinates Laplace’s equation becomes

\begin{equation}
u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta \theta} = 0
\end{equation}

We’ll now apply Separation of Variables to this PDE. Setting $u(r, \theta) = R(r)T(\theta)$ and plugging in we get

\[
R''T + \frac{R'}{r} T + \frac{R}{r^2} T'' = 0
\]

Multiplying both sides by $r^2 / RT$ we get

\[
\frac{r^2}{R} R'' + \frac{r}{R} R' = - \frac{T''}{T}
\]

Observing that each side depends only a variable that does not appear on the opposite side we conclude that both sides must be equal to a constant. Let’s denote this constant by $\lambda^2$. We then have

\[
\frac{r^2}{R} R'' + \frac{r}{R} R' = \lambda^2 = - \frac{T''}{T}
\]
or the following pair of ordinary differential equations

\[(4a)\]  \[T'' = -\lambda^2 T\]
\[(4b)\]  \[r^2 R'' + r R' = \lambda^2 R\]

The first equation \((4a)\) should be quite familiar by now. It has as its general solution

\[(5)\]  \[T(\theta) = A \cos (\lambda \theta) + B \sin (\lambda \theta)\]

The second equation \((4b)\) is an Euler type equation. Such equations can (almost always) be solved using the ansatz \(R(r) = r^m\), regarding \(m\) as an adjustable parameter. Inserting \(r^m\) in place of \(R\) in \((4b)\) we get

\[0 = r^2 \left( m(m - 1)r^{m-2} \right) + r \left( mr^{m-1} \right) - \lambda^2 \left( r^m \right) = (m(m - 1) + m - n^2) r^m = (m^2 - \lambda^2) r^m\]

Evidently, we must take \(m^2 = \lambda^2\) which in turn requires \(m = \pm \lambda\). The general solution of \((4b)\) is thus

\[(6)\]  \[R(r) = ar^\lambda + br^{-\lambda}\]

Err, except that something screwy happens when \(\lambda = 0\); in that case we only get one linearly independent solution \(R(r) = a\) some constant. To get a second linearly independent solution for the \(\lambda = 0\) case, we can employ Reduction of Order, or simply solve

\[(7)\]  \[r^2 R'' + r R' = 0\]

afresh. The latter is easy enough. Dividing out by \(r\) and setting \(S = R', (7)\) becomes

\[
\frac{r S' + S}{S} = 0 \implies \frac{1}{S} \frac{dS}{dr} = -r \implies \frac{dS}{S} = -\frac{dr}{r} \implies \int \frac{dS}{S} = -\int \frac{dr}{r} + C
\]

\[
\implies \ln |S| = -\ln |r| + C
\]

Setting \(C = \ln |c|\) we then have

\[
\ln |S| = -\ln |r| + \ln |c| = \ln \left| \frac{c}{r} \right| \implies S = \frac{c}{r}
\]

Finally, we integrate \(S\) to recover \(R\)

\[
R(r) = \int R' dr + d = \int S dr + d = \int c r^{-\lambda} dr + d = c \ln |r| + d \quad (\lambda = 0 \text{ case})
\]

In conclusion, the general solution of \((4b)\) is

\[(8)\]  \[R(r) = \begin{cases} cr^\lambda + dr^{-\lambda} & \text{if } \lambda \neq 0 \\ c \ln |r| + d & \text{if } \lambda = 0 \end{cases}\]

Putting \((5)\) and \((8)\) together, we obtain

\[
u_\lambda (r, \theta) = \begin{cases} (A_\lambda \cos (\lambda \theta) + B_\lambda \sin (\lambda \theta)) r^\lambda + (C_\lambda \cos (\lambda \theta) + D_\lambda \sin (\lambda \theta)) r^{-\lambda} & \lambda \neq 0 \\ A_0 + C_0 \ln |r| & \lambda = 0 \end{cases}
\]

as Separation-of-Variables-type solutions to Laplace’s equation in polar coordinates (n.b. when \(\lambda = 0\), \(\cos (\lambda \theta) = 1\) and \(\sin (\lambda \theta) = 0\)).

Now we can whittle down this set of possible solutions even further by imposing some hidden boundary conditions (besides \((2b)\)).

One thing we expect of any viable solution is that \(u(r, \theta) = u(r, \theta + 2\pi)\), for after all a point with coordinates \([r \cos (\theta), r \sin (\theta)]\) is the same as the point with coordinates \([r \cos (\theta + 2\pi), r \sin (\theta + 2\pi)]\). But

\[
\begin{align*}
\cos (\lambda \theta) &= \cos (\lambda (\theta + 2\pi)) \\
\sin (\lambda \theta) &= \sin (\lambda (\theta + 2\pi))
\end{align*}
\]

\[
\iff \lambda = n \in \mathbb{Z} := \{0, \pm 1, \pm 2, \pm 3, \ldots\}
\]

That is, \(\lambda\) must be an integer, we may as well take to be nonnegative since the solutions where \(\lambda = n\) and \(\lambda = -n\) are actually coincide (up to changing the signs of the arbitrary constants \(B\) and \(D\)).
Secondly, we expect any viable solution to be continuous at \( r = 0 \). This will require us to throw out the solutions where \( C \) and \( D \) are non-zero; for both \( r^{-n} \) and \( \ln |r| \) become unbounded as \( r \to 0 \). We thus have

\[
u_n (r, \theta) = \begin{cases} A_n \cos (n\theta) r^n + B_n \sin (n\theta) r^n & \text{if } n = 1, 2, 3, \ldots \\ A_0 & \text{if } n = 0 \end{cases}
\]

or even more succinctly

(9) \[ u_n (r, \theta) = A_n \cos (n\theta) r^n + B_n \sin (n\theta) r^n, \quad n = 0, 1, 2, 3, \ldots \]

Now let’s take a general linear combination of the solutions (9) and impose the boundary condition (2b). Thus, we set

\[ u (r, \theta) = \sum_{n=0}^{\infty} A_n \cos (n\theta) r^n + \sum_{n=1}^{\infty} B_n \sin (n\theta) r^n \]

and impose

(10) \[ h (\theta) = u (a, \theta) = \sum_{n=0}^{\infty} A_n \cos (n\theta) a^n + \sum_{n=1}^{\infty} B_n \sin (n\theta) a^n \]

Now from Fourier theory we know that any continuous function on the circle has a unique Fourier expansion whose coefficients can be explicitly determined in terms of certain Fourier integrals. Applying this theory to the case at hand we have

(11a) \[ h (\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos (n\theta) + \sum_{n=1}^{\infty} b_n \sin (n\theta) \]

(11b) \[ a_n = \frac{1}{\pi} \int_{0}^{2\pi} h (\theta) \cos (n\theta) \, d\theta \]

(11c) \[ b_n = \frac{1}{\pi} \int_{0}^{2\pi} h (\theta) \sin (n\theta) \, d\theta \]

Comparing (10) with (11a) we conclude that our solution must be

\[ u (r, \theta) = A_0 + \sum_{n=1}^{\infty} A_n \cos (n\theta) r^n + \sum_{n=1}^{\infty} B_n \sin (n\theta) r^n \]

with

\[ A_0 = \frac{a_0}{2} = \frac{1}{2\pi} \int_{0}^{2\pi} h (\theta) \, d\theta \]

\[ A_n = \frac{1}{a^n \pi} \int_{0}^{2\pi} h (\theta) \cos (n\theta) \, d\theta, \quad n = 1, 2, 3, \ldots \]

\[ B_n = \frac{1}{a^n \pi} \int_{0}^{2\pi} h (\theta) \sin (n\theta) \, d\theta, \quad n = 1, 2, 3, \ldots \]
1. Poisson Sum Formula

We now have

\[ u(r, \theta) = A_0 + \sum_{n=1}^{\infty} A_n \cos(n\theta) r^n + \sum_{n=1}^{\infty} B_n \sin(n\theta) r^n \]

\[ = \frac{1}{2\pi} \int_0^{2\pi} h(\theta) d\theta \]

\[ + \sum_{n=1}^{\infty} \frac{1}{a^2 \pi} \int_0^{2\pi} h(\theta') \cos(n\theta') \cos(n\theta) d\theta' r^n \]

\[ + \sum_{n=1}^{\infty} \frac{1}{a^2 \pi} \int_0^{2\pi} h(\theta') \sin(n\theta') \sin(n\theta) d\theta' r^n \]

or, interchanging the order of summation and integration (which we can do so long as the series converges in the first place)

\[ u(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} h(\theta') \left( 1 + 2 \sum_{n=1}^{\infty} (\cos(n\theta) \cos(n\theta') + \sin(n\theta) \sin(n\theta')) \left( \frac{r}{a} \right)^n \right) d\theta' \]

(13)

\[ = \frac{1}{2\pi} \int_0^{2\pi} h(\theta') \left( 1 + 2 \sum_{n=1}^{\infty} \cos(n(\theta - \theta')) \left( \frac{r}{a} \right)^n \right) d\theta \]

where we have employed the cosine angle sum formula

\[ \cos(\alpha - \beta) = \cos(\alpha) \cos(\beta) + \sin(\alpha) \sin(\beta) \]

Next, inserting

\[ \cos(\phi) = \frac{e^{i\phi} + e^{-i\phi}}{2} \]

into the series

(15)

\[ 1 + 2 \sum_{n=1}^{\infty} \cos(n\phi) t^n \]

we get

\[ 1 + \sum_{n=1}^{\infty} (e^{i\phi} + e^{-i\phi}) t^n = 1 + \sum_{n=1}^{\infty} (e^{i\phi} t)^n + \sum_{n=1}^{\infty} (e^{-i\phi} t)^n \]

Using the well-known geometric series

\[ \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \]

we obtain

\[ 1 + 2 \sum_{n=1}^{\infty} \cos(n\phi) t^n = 1 + \left( \frac{1}{1 - e^{i\phi} t} - 1 \right) + \left( \frac{1}{1 - e^{-i\phi} t} - 1 \right) \]

\[ = \frac{1 - t^2}{1 - e^{i\phi} t - e^{-i\phi} t + t^2} \]

\[ = \frac{1 - t^2}{1 - 2 \cos \phi + t^2} \]

or

(18)

\[ 1 + 2 \sum_{n=1}^{\infty} \cos(n\phi) t^n = \frac{1 - t^2}{1 - 2 \cos \phi + t^2} \]
Applying this last formula to
\[ \left(1 + 2 \sum_{n=1}^{\infty} \cos (n (\theta - \theta')) \left( \frac{r}{a} \right)^n \right) \]
we find
\[ \left(1 + 2 \sum_{n=1}^{\infty} \cos (n (\theta - \theta')) \left( \frac{r}{a} \right)^n \right) = \frac{1 - (\frac{r}{a})^2}{1 - 2 \cos (\theta - \theta') \left( \frac{r}{a} \right)^2} = \frac{a^2 - r^2}{a^2 - \cos (\theta - \theta') ar + r^2} \]
Thus,
\[ (20) \quad u (r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} h (\theta') \frac{a^2 - r^2}{a^2 - 2 \cos (\theta - \theta') ar + r^2} d\theta' \]
Finally, we’ll convert this result into something that’s understandable in a coordinate free way. Set
\[ x = [r \cos (\theta), r \sin (\theta)] \]
\[ y = [a \cos (\theta'), a \sin (\theta')] \]
then
\[ \|x\| = r^2 \]
\[ \|y\| = a^2 \]
and
\[ \|x - y\|^2 = (r \cos \theta - a \cos \theta')^2 + (r \sin \theta - a \sin \theta')^2 \]
\[ = r^2 \cos^2 \theta - 2ar \cos \theta \cos \theta' + a^2 \cos^2 \theta' \]
\[ + r^2 \sin^2 \theta - 2ar \sin \theta \sin \theta' + a^2 \sin^2 \theta' \]
\[ = r^2 (\cos^2 \theta + \sin^2 \theta) - 2ar (\cos \theta \cos \theta' + \sin \theta \sin \theta') + a^2 (\cos^2 \theta' + \sin^2 \theta') \]
\[ = r^2 - 2ar \cos (\theta - \theta') + a^2 \]
and so we can write
\[ (21) \quad u (x) = \frac{1}{2\pi a} \int_{y \in \partial D} h (y) \frac{a^2 - \|x\|^2}{\|x - y\|^2} ds \]
where \(\partial D\) is the circle bounding \(D\). The additional factor of \(1/a\) arises because
\[ ad\theta = ds \]
is the correct expression for the infinitesimal arc-length when we interprete (20) as a path integral about the boundary of \(D\).