Separation of Variables and Fourier Series

Recall that in Lecture 5 we constructed the general solution of the Wave Equation with Cauchy boundary conditions

\[
\begin{align*}
  u_{tt} - c^2 u_{xx} &= 0 \quad , \quad -\infty < x < +\infty \quad , \quad t > 0 \\
  u(x,0) &= \phi(x) \quad , \quad -\infty < x < +\infty \\
  u_t(x,0) &= \psi(x) \quad , \quad -\infty < x < +\infty 
\end{align*}
\]  

(1a) (1b) (1c)

that was valid in situation where \(-\infty < x < +\infty\). This lead to

\[
u(x,t) = \frac{1}{2} (\phi(x + ct) + \phi(x - ct)) + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(\tau) d\tau
\]

as the unique solution.

Then in Lecture 6, we constructed the general solution of the corresponding problem on the half-line

\[
\begin{align*}
  u_{tt} - c^2 u_{xx} &= 0 \quad , \quad 0 < x < +\infty \quad , \quad t > 0 \\
  u(x,0) &= \phi(x) \quad , \quad 0 < x < +\infty \\
  u_t(x,0) &= \psi(x) \quad , \quad 0 < x < +\infty \\
  u(0,t) &= 0 \quad , \quad t > 0 
\end{align*}
\]  

(2a) (2b) (2c) (2d)

The solution of (2) was obtained by extending the domain \(0 < x < +\infty\) to the whole line (simultaneously extending \(\phi(x)\) and \(\psi(x)\) to odd functions on the whole real line) and then observing that the restriction of the solution of the extended problem

\[
u(x,t) = \frac{1}{2} \phi_{\text{odd}}(x + ct) + \phi_{\text{odd}}(x - ct)) + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi_{\text{odd}}(\tau) d\tau
\]

to the half-line \(0 \leq x < +\infty\) would satisfy each of the conditions in (2).

In this lecture we will consider the wave-equation on the finite interval \(0 \leq x \leq L\): viz.,

\[
\begin{align*}
  u_{tt} - c^2 u_{xx} &= 0 \quad , \quad 0 < x < L \quad , \quad t > 0 \\
  u(x,0) &= \phi(x) \quad , \quad 0 < x < L \\
  u_t(x,0) &= \psi(x) \quad , \quad 0 < x < L \\
  u(0,t) &= 0 \quad , \quad t > 0 \\
  u(L,t) &= 0 \quad , \quad t > 0
\end{align*}
\]  

(3a) (3b) (3c) (3d) (3e)

This finite-interval problem can also be solved by finding a suitable extension to a problem on the entire real line, solving it there, and then restricting the answer back to the interval \([0,L]\); but the resulting answer turns out to be rather messy (see pg. 78 of the text). In fact, the essential messiness of such a solution can be inferred from the messiness is the corresponding physical situation: an initial wave would split into two waves of the same shape but half the size moving in opposite directions, both of these wave would then propagate towards one of the end points of the interval, bounce off the endpoints, come back upside-down, pass through one another, then bounce off the opposite end point, come back right-side-up, pass through one another, .... etc.
Today we introduce a different approach to the solution of (3).

1. Separation of Variables

The basic idea behind our method is the same as usual: find a solution, generalize it to get more solutions, until we have enough solutions to form a solution that actually satisfies the boundary boundary conditions.

We’ll start by looking for a solution of

\[
\begin{align*}
\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} &= 0 \quad (4a) \\
u (0, t) &= 0 \quad (4b) \\
u (0, L) &= 0 \quad (4c)
\end{align*}
\]

of the form

\[u (x, t) = X (x) T (t)
\]

Plugging this expression into the wave equation (4a) we find

\[
u_{tt} - c^2 u_{xx} = 0 \quad \implies \quad XT'' - c^2 TX'' = 0
\]

or

\[
\frac{1}{c^2} \frac{T''}{T} = \frac{X''}{X}
\]

(6)

We then argue that since (6) must be true for all \(x\) and all \(t\), and because the left hand side of (6) doesn’t depend on \(x\) while the right hand side doesn’t depend on \(t\), both sides must be equal to a constant independent of \(x\) and \(t\). Thus, if (5) is to satisfy (4), there must be a constant \(\Lambda\) such that

\[
\frac{1}{c^2} \frac{T''}{T} = \Lambda = \frac{X''}{X}
\]

and, moreover, \(T (t)\) and \(X (x)\) must satisfy ordinary differential equations of the form

\[
\begin{align*}
T'' &= c^2 \Lambda T \\
X'' &= \Lambda X
\end{align*}
\]

(7)

Now at this point we haven’t assumed anything about the constant \(\Lambda\) – it could be a positive, negative, or even a complex number. What I’ll show next is the boundary conditions (4b) and (4c) reduce these possibilities tremendously.

Ok. Suppose \(\lambda\) is one of the square roots of \(\Lambda\) (N.B. even if \(\Lambda = re^{i\theta}\) is complex, it will have a square root \(\lambda = \sqrt{\lambda}e^{i\theta/2}\)). The general solution of an ODE of the form

\[
y'' = \lambda^2 y
\]

is

\[
y = c_1 e^{\lambda x} + c_2 e^{-\lambda x}
\]

And so in order to satisfy (7) (with \(\Lambda = \lambda^2\)) \(T (t)\) and \(X (t)\) must be functions of the form

\[
\begin{align*}
T (t) &= a_1 e^{\lambda c t} + a_2 e^{-\lambda c t} \\
X (t) &= b_1 e^{\lambda x} + b_2 e^{-\lambda x}
\end{align*}
\]

In addition, we want the \(u (x, t) = T (t) X (x)\) to satisfy (4b) and (4c). So

\[
\begin{align*}
0 &= u (0, t) = (a_1 e^{\lambda c t} + a_2 e^{-\lambda c t}) (b_1 + b_2) \\
0 &= u (L, t) = (a_1 e^{\lambda c t} + a_2 e^{-\lambda c t}) (b_1 e^{\lambda L} + b_2 e^{-\lambda L})
\end{align*}
\]

Now the functions \(e^{\lambda c t}\) and \(e^{-\lambda c t}\) are linearly independent functions of \(t\). So the first factors can not be expected to vanish for all \(t\) unless \(a_1 = a_2 = 0\), which would imply that \(T (t) = 0\) and hence \(u (x, t) = T (t) X (x) = 0\) for all \(x\) and \(t\). We’ll ignore this trivial solution.
So we’ll need

\[ b_1 + b_2 = 0 \]
\[ b_1 e^{\lambda L} + b_2 e^{-\lambda L} = 0 \]

The first equation allows us to eliminate \( b_2 \) in favor of \( b_1 \) and then the second equation implies

\[ b_1 e^{\lambda L} - b_1 e^{-\lambda L} = 0 \implies b_1 (e^{\lambda L} - e^{-\lambda L}) = 0 \]

Again we have a trivial solution \( b_1 = 0 \). And so we are reduced to looking at

\[ e^{\lambda L} = e^{-\lambda L} \]

Now write

\[ \lambda = \text{Re} \left( \lambda \right) + i \text{Im} \left( \lambda \right) = \alpha + i\beta \]

Equation (8) becomes

\[ e^{(\alpha + i\beta)L} = e^{-(\alpha + i\beta)L} \]

Now the modulus \( |z| = \sqrt{2} \) of the left hand side is \( e^{\alpha L} \), while that of the left is \( e^{-\alpha L} \). These cannot be equal unless \( \alpha = 0 \). So we conclude that the real part of \( \lambda \) must be zero. So now we require

\[ e^{i\beta L} = e^{-i\beta L} \]

or

\[ 0 = e^{i\beta L} - e^{-i\beta L} = 2i \sin (\beta L) \implies \beta L = 0, \pm \pi, \pm 2\pi, \pm 3\pi, \ldots \]

We conclude that

\[ \beta = \pm \frac{n\pi}{L}, \quad n = 0, 1, 2, \ldots \]

which implies

\[ \lambda = \pm i \left( \frac{n\pi}{L} \right) \implies \Lambda = \lambda^2 = -\frac{n^2\pi^2}{L^2}, \quad n = 0, 1, 2, \ldots \]

and, moreover,

\[ X(t) = b_1 e^{\pm \frac{n\pi x}{L}} - b_1 e^{\mp \frac{n\pi x}{L}} = \pm 2ib_1 \sin \left( \frac{n\pi}{L} x \right) \]

\[ T(t) = a_1 e^{\pm \frac{n\pi c t}{L}} + a_2 e^{\mp \frac{n\pi c t}{L}} \]

In summary, in looking for a solution of

\[ u_{tt} - c^2 u_{xx} = 0 \]

\[ u(0,t) = 0 \]

\[ u(0,L) = 0 \]

or the form

\[ u(x,t) = X(x) T(t) \]

we have found, not just one, but infinitely many solutions of the form

\[ u_n(x,t) = a_n \cos \left( \frac{n\pi c}{L} t \right) \sin \left( \frac{n\pi x}{L} \right) + b_n \sin \left( \frac{n\pi c}{L} t \right) \sin \left( \frac{n\pi x}{L} \right), \quad n = 0, 1, 2, \ldots \]

The next step in our program will be form a linear combination of such solutions

\[ u(x,t) = \sum_{n=1}^{\infty} a_n \cos \left( \frac{n\pi c}{L} t \right) \sin \left( \frac{n\pi x}{L} \right) + \sum_{n=1}^{\infty} b_n \sin \left( \frac{n\pi c}{L} t \right) \sin \left( \frac{n\pi x}{L} \right) \]
and see if by a suitable choice of the coefficients \( a_n \) and \( b_n \) we can get the obtain a function solving

\[
\begin{align*}
    u_{tt} - c^2 u_{xx} &= 0, \quad 0 < x < L, \quad t > 0 \\
    u(x, 0) &= \phi(x), \quad 0 < x < L \\
    u_t(x, 0) &= \psi(x), \quad 0 < x < L \\
    u(0, t) &= 0, \quad t > 0 \\
    u(L, t) &= 0, \quad t > 0
\end{align*}
\]

(3a)
(3b)
(3c)
(3d)
(3e)

Now, in fact, because the PDE (3a) is linear and homogeneous, (7) or any linear combination of the solutions (6) will automatically satisfy (3a). Also since each \( u_n(x, t) \) will satisfy (3d) and (3e), so too will any linear combination.

It remains to impose (3b) and (3c). These lead to

\[
\begin{align*}
    \phi(x) &= u(x, 0) = \sum_{n=1}^{\infty} a_n \cos \left( \frac{n\pi c}{L} t \right) \sin \left( \frac{n\pi x}{L} \right) \bigg|_{t=0} \\
    &\quad + \sum_{n=1}^{\infty} a_n \sum_{n=1}^{\infty} b_n \sin \left( \frac{n\pi c}{L} t \right) \sin \left( \frac{n\pi x}{L} \right) \\
    \psi(x) &= u_t(x, 0) = \sum_{n=1}^{\infty} \left( -\frac{n\pi c}{L} \right) \sin \left( \frac{n\pi c}{L} t \right) \sin \left( \frac{n\pi x}{L} \right) \bigg|_{t=0} \\
    &\quad + \sum_{n=1}^{\infty} b_n \left( \frac{n\pi c}{L} \right) \cos \left( \frac{n\pi c}{L} t \right) \sin \left( \frac{n\pi x}{L} \right) \\
\end{align*}
\]

or

\[
\begin{align*}
    \phi(x) &= \sum_{n=1}^{\infty} a_n \sin \left( \frac{n\pi x}{L} \right) \\
    \psi(x) &= \sum_{n=1}^{\infty} b_n \left( \frac{n\pi c}{L} \right) \sin \left( \frac{n\pi x}{L} \right)
\end{align*}
\]

This might seem like an extremely complicated system of equations to solve. What we’ll do next time is develop the theory of Fourier series which will in turn allow us to solve such equations quite easily.