Second Order Linear Equations

1. The Basic Types of 2\textsuperscript{nd} Order Linear PDEs:

1.1. Generic and Standard Forms of 2\textsuperscript{nd} Order Linear PDEs. The generic form of a second order linear PDE in two variables is

$$(1)
A(x, y) \frac{\partial^2 \phi}{\partial x^2} + B(x, y) \frac{\partial^2 \phi}{\partial x \partial y} + C(x, y) \frac{\partial^2 \phi}{\partial y^2} + D(x, y) \frac{\partial \phi}{\partial x} + E(x, y) \frac{\partial \phi}{\partial y} + F(x, y) \phi = G(x, y)
$$

We shall see in a second that by a suitable change of coordinates \(x, y \rightarrow \xi(x, y), \eta(x, y)\) we can cast any PDE of the form (1) into one of the following three (standard) forms.

(P) Parabolic Equations:

$$(2)
\frac{\partial^2 \Phi}{\partial \xi^2} + f_1(\xi, \eta) \frac{\partial \Phi}{\partial \xi} + f_2(\xi, \eta) \frac{\partial \Phi}{\partial \eta} + f_3(\xi, \eta) \Phi = g(\xi, \eta)
$$

(E) Elliptic Equations:

$$(3)
\frac{\partial^2 \Phi}{\partial \xi^2} + \frac{\partial^2 \Phi}{\partial \eta^2} + f_1(\xi, \eta) \frac{\partial \Phi}{\partial \xi} + f_2(\xi, \eta) \frac{\partial \Phi}{\partial \eta} + f_3(\xi, \eta) \Phi = g(\xi, \eta)
$$

(H) Hyperbolic Equations:

$$(4)
\frac{\partial^2 \Phi}{\partial \xi \partial \eta} + f_1(\xi, \eta) \frac{\partial \Phi}{\partial \xi} + f_2(\xi, \eta) \frac{\partial \Phi}{\partial \eta} + f_3(\xi, \eta) \Phi = g(\xi, \eta)
$$

To show this, it is helpful to rewrite things in a matrix notation. First let’s introduce an index notation for our coordinate systems by letting \(x \rightarrow x_1, y \rightarrow y_1, \xi \rightarrow \xi_1, \eta \rightarrow \xi_2\) and letting

\[
A_{11}(x) = A_{11}(x_1, x_2) = A(x, y) \\
A_{12}(x) = A_{12}(x) = \frac{1}{2} B(x, y) \\
A_{22}(x) = C(x, y)
\]

so that the leading terms (the terms with the second order derivatives) of our PDE correspond to the differential operator.

\[
\mathcal{L}_2 = \sum_{i,j} A_{i,j}(x) \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j}
\]

We would like to find a change of variables that simplifies this differential operator as much as possible. Now let us introduce a change of coordinates where the new coordinates \(\xi = [\xi_1, \xi_2]\) are expressed as certain functions of the old coordinates

\[
\xi_2 = \xi_1(x_1, x_2) \\
\xi_2 = \xi_2(x_1, x_2)
\]
1. THE BASIC TYPES OF 2nd ORDER LINEAR PDES:

Now the Chain Rule gives us a rule for constructing the differential operator $\mathcal{L}_2$ with respect to the new variables that corresponds to the action of the original differential operator $\mathcal{L}_2$. Indeed, the Chain Rule says that

$$\frac{\partial}{\partial x_i} = \sum_a \frac{\partial \xi_a}{\partial x_i} \frac{\partial}{\partial \xi_a}$$

and so

$$\mathcal{L}_2 = \sum_{i,j} A_{i,j}(x) \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} = \sum_{i,j} A_{ij}(x) \sum_{a,b} \frac{\partial \xi_a}{\partial x_i} \frac{\partial \xi_b}{\partial \xi_j} \frac{\partial}{\partial \xi_a} \frac{\partial}{\partial \xi_b}$$

$$= \sum_{a,b} \sum_{i,j} \frac{\partial \xi_a}{\partial x_i} A_{ij}(x) \frac{\partial \xi_b}{\partial \xi_j} \frac{\partial}{\partial \xi_a} \frac{\partial}{\partial \xi_b} + \text{lower order terms}$$

So, in terms of the new variable the second order terms of the differential equation correspond to the differential operator

$$\tilde{\mathcal{L}}_2 = \sum_{a,b} \sum_{i,j} \frac{\partial \xi_a}{\partial x_i} A_{ij}(x) \frac{\partial \xi_b}{\partial \xi_j} \frac{\partial}{\partial \xi_a} \frac{\partial}{\partial \xi_b}$$

Let us again introduce some more matrix notation by defining a $2 \times 2$ matrix $J$ as

$$J_{ia} = \left( \frac{\partial \xi_i}{\partial x_a} \right)_{i,a} = \frac{\partial \xi_a}{\partial x_i}, \quad i, a = 1, 2$$

(We note that the absolute value of the determinant of this matrix is the Jacobian of the coordinate transformation). This then allows us to write $\tilde{\mathcal{L}}_2$ as

$$\tilde{\mathcal{L}}_2 = \sum_{a,b} (JAJ)_{ab} \frac{\partial}{\partial \xi_a} \frac{\partial}{\partial \xi_b}$$

In short, after making a change of coordinates the coefficient matrix of the second order terms is related to the original coefficient matrix $A$ by

$$A \to JAJ$$

where

$$J_{ia} = \frac{\partial \xi_a}{\partial x_i}$$

We now quote a theorem from linear algebra:

**Theorem 4.1.** If $A$ is a real symmetric matrix, there is an orthogonal matrix $O$ such that $OAO$ is a diagonal matrix.

Thus, by now choosing our coordinate transformation so that the matrix $J$ corresponds to a suitable orthogonal matrix, we can send the matrix $A$ to a diagonal matrix:

$$A \to JAJ$$

In fact, with a little more work, one can find coordinate transformations so that $JAJ$ takes one of the following three forms

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

To understand, why we should have at least these three case, we note that the if $J$ is an orthogonal matrix $A$ has the same determinant as $JAJ$. The three matrices above have determinants of, respectively, 1, $-1$ and 0. These three cases correspond to the situations where the original matrix $A$ has a positive, negative or zero determinant.

So let me state clearly the general situation for second order linear PDEs.
Proposition 4.2. If

\[ \sum_{i,j} A(x)_{ij} \frac{\partial^2 \phi}{\partial x_i \partial x_j} + \sum_i B(x)_i \frac{\partial \phi}{\partial x_i} + C(x) \phi = D(x) \]

is a second order linear PDE, there are three possibilities depending on the sign of the determinant of the matrix \( A(x) \):

Elliptic Case: If \( \det A(x) > 0 \), there is a coordinate transformation \( x \to \xi \) that sends the PDE to a PDE of the form

\[ \frac{\partial^2 \Phi}{\partial \xi^2} + \frac{\partial^2 \Phi}{\partial \eta^2} + f_1(\xi, \eta) \frac{\partial \Phi}{\partial \xi} + f_2(\xi, \eta) \frac{\partial \Phi}{\partial \eta} + f_3(\xi, \eta) \Phi = g(\xi, \eta) \]

Hyperbolic Case: If \( \det A(x) < 0 \), there is a coordinate transformation \( x \to \xi \) that sends the PDE to a PDE of the form

\[ \frac{\partial^2 \Phi}{\partial \xi^2} - \frac{\partial^2 \Phi}{\partial \eta^2} + f_1(\xi, \eta) \frac{\partial \Phi}{\partial \xi} + f_2(\xi, \eta) \frac{\partial \Phi}{\partial \eta} + f_3(\xi, \eta) \Phi = g(\xi, \eta) \]

Parabolic Case: If \( \det A(x) > 0 \), there is a coordinate transformation \( x \to \xi \) that sends the PDE to a PDE of the form

\[ \frac{\partial^2 \Phi}{\partial \xi^2} + f_1(\xi, \eta) \frac{\partial \Phi}{\partial \xi} + f_2(\xi, \eta) \frac{\partial \Phi}{\partial \eta} + f_3(\xi, \eta) \Phi = g(\xi, \eta) \]

2. The Basic Prototypes

Associated to each of these standard forms are prototypical examples, each of which also corresponds to a fundamental PDE occurring in physical applications. For the next few weeks we shall discuss the solutions or each of these equations extensively.

2.1. The Heat Equation.

(5) \[ \frac{\partial \phi}{\partial t} - a^2 \frac{\partial^2 \phi}{\partial x^2} = 0 \]

This equation arises in studies of heat flow. For example, if a 1-dimensional wire is heated at one end, then the function \( \phi(x, t) \) describing the temperature of the wire at position \( x \) and time \( t \) will satisfy (5). The heat equation is the prototypical example of a parabolic PDE.

2.2. Laplace’s Equation.

(6) \[ \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \]

This equation arises in a variety of physical situations: the function \( \phi(x, y) \) might be interpretable as the electric potential at a point \( (x, y) \) in the plane, or the steady state temperature of a point in the plane. Laplace’s equation is the prototypical example of an elliptic PDE.

2.3. The Wave Equation.

(7) \[ \frac{\partial^2 \phi}{\partial t^2} - a^2 \frac{\partial^2 \phi}{\partial x^2} = 0 \]

This equation governs the propagation of waves in a medium, such as the vibrations of a taunt string, pressure fluctuations in a compressible fluid, or electromagnetic waves. The wave equation is the prototypical example of a hyperbolic PDE. The coordinate transformation that casts (7) into the form (4) is

\[ \xi = x - ct \]
\[ \eta = x + ct \]
3. Boundary Conditions

In stark contrast to the theory of ordinary differential equations where boundary conditions play a relatively innocuous role in the construction of solutions, the nature of the boundary conditions imposed on a partial differential equation can have a dramatic effect on whether or not the PDE/BVP (partial differential equation / boundary value problem) is solvable.

There are three particular kinds of boundary conditions that are particularly common in physical applications.

3.1. Cauchy Conditions. Suppose $L[\phi] = G(x, \phi)$ is a PDE imposed on a region $R \subset \mathbb{R}^n$ with boundary $\partial R$. Cauchy boundary conditions in such a situation would be the specification of the function and its normal derivative along the boundary curve.

![Cauchy Boundary Conditions Diagram](image)

Cauchy boundary conditions are commonly applicable in dynamical situations (where the system is interpreted as evolving with respect to a time parameter $t$).

3.2. Dirichlet Conditions. The specification of the function on the boundary curve.

![Dirichlet Boundary Conditions Diagram](image)

As an example of a PDE/BVP with Dirichlet boundary conditions, consider the problem of finding the equilibrium temperature distribution of a rectangular sheet whose edges are maintained at some prescribed (but non-constant) temperature.

3.3. Neumann Conditions. The specification of the normal derivative of the function along the boundary curve.
As an example of a PDE/BVP with Neumann boundary conditions, consider the problem of determining the electric potential inside a superconducting cylinder.