LECTURE 1

Introduction

1. Partial Differential Equations: Basics

A partial differential equation (or PDE, for short) is an equation relating a function \( \phi \) of \( n \) variables \( x_1, \ldots, x_n \), its partial derivatives with respect to the variables \( x_1, \ldots, x_n \), and the variables themselves; that is to say, an equation of the form

\[
F \left[ \phi, x_1, \ldots, x_n, \frac{\partial \phi}{\partial x_1}, \ldots, \frac{\partial \phi}{\partial x_n}, \frac{\partial^2 \phi}{\partial x_1 \partial x_2}, \ldots \right] (x) = 0
\]

The order of the partial differential equation is the order of the highest derivative appearing in the PDE.

If the dependence of the functional \( F \) on \( \phi \) and its partial derivatives is linear, then the PDE (1) is said to be **linear**. In this introductory course we shall concentrate primarily on linear PDEs.

**Remark 1.1.** We might point out that there are several common conventions for expressing partial derivatives. First of all, it is especially common in physical applications to denote the underlying variables as \( x, y, z \) (or perhaps some other letters) rather than \( x_1, x_2, x_3 \). Secondly, it is common to employ following shorthand expressions for partial derivatives

\[
\frac{\partial \phi}{\partial x} = \phi_x
\]

\[
\frac{\partial^2 \phi}{\partial x \partial y} = \phi_{xy}
\]

etc.

In these notes, we shall often pass back and forth between these various notations without comment.

2. Linear PDEs

A function \( f \) of one-variable \( x \) is **linear** if it can be expressed in the form

\[ f(x) = ax + b \]

More generally, a function \( F \) of several variables \( x_1, \ldots, x_i \) if can be expressed as

\[ F(x_1, \ldots, x_n) = A_1(x_{i+1}, \ldots, x_n)x_1 + A_2(x_{i+1} + \cdots + x_n)x_2 + \cdots + A_i(x_{i+1}, \ldots, x_n)x_i + B(x_{i+1}, \ldots, x_n) \]

or put another way

\[
\frac{\partial}{\partial x_k} \frac{\partial}{\partial x_\ell} F = 0 \quad \text{for all } k, \ell \in \{1, \ldots, i\}
\]

**Definition 1.2.** A partial differential equation

\[
F \left[ \phi, x_1, \ldots, x_n, \frac{\partial \phi}{\partial x_1}, \ldots, \frac{\partial \phi}{\partial x_n}, \frac{\partial^2 \phi}{\partial x_1 \partial x_2}, \ldots \right] = 0
\]

is said to be **linear** if the function \( F \) that defines it is simultaneously linear with respect to \( \phi, \frac{\partial \phi}{\partial x_1}, \ldots, \frac{\partial^2 \phi}{\partial x_1 \partial x_2}, \ldots \)
N.B. A linear partial differential equation may be non-linear with respect to the underlying variables \( x_1, \ldots, x_n \); however, it must be simultaneously linear with respect to the unknown function \( \phi \) and all of its partial derivatives. In other words, a linear PDE will have the form

\[
0 = A_0 (x_1, \ldots, x_n) \phi + \sum_{i=1}^{n} A_i (x_1, \ldots, x_n) \frac{\partial \phi}{\partial x_i} + \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} (x_1, \ldots, x_n) \frac{\partial^2 \phi}{\partial x_i \partial x_j} + \cdots
\]

### 2.1. Differential operators.

Linear PDEs are often expressed in terms of differential operators. This means thinking of the function \( F \) that defines the PDE as a sum of terms of the form

\[
A_{i_1 \cdots i_k} (x_1, \ldots, x_n) \frac{\partial}{\partial x_{i_1}} \cdots \frac{\partial}{\partial x_{i_k}}
\]

and then regarding such terms as arising via the application of the “monomial differential operator” of the form

\[
A_{i_1 \cdots i_k} (x_1, \ldots, x_n) \frac{\partial}{\partial x_{i_1}} \cdots \frac{\partial}{\partial x_{i_k}}
\]

to the function \( \phi \). The differential operator corresponding to \( F \) is then just the formal sum of the differential operators of the individual terms appearing in \( F \).

**Example 1.3.**

\[
\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + xy = 0
\]

is a linear PDE of degree 2. In differential operator notation, we might express this PDE as

\[
\mathcal{L} [\phi] = 0
\]

where \( \mathcal{L} \) is the differential operator defined by

\[
\mathcal{L} [\phi] := \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + xy \right) \phi
\]

**Example 1.4.**

\[
x_1 x_2 \frac{\partial^2 \phi}{\partial x_1^2} + x_2 \frac{\partial \phi}{\partial x_2} + x_1^2 \phi - x_1^2 - x_2^2 = 0
\]

is a linear PDE of degree 2. If we define the differential operator \( \mathcal{L} \) to be the operation that sends a differentiable function \( \phi \) to

\[
\mathcal{L} [\phi] := x_1 x_2 \frac{\partial^2 \phi}{\partial x_1^2} + x_2 \frac{\partial \phi}{\partial x_2} + x_1^2 \phi
\]

In this case, one would write

\[
\mathcal{L} = x_1 x_2 \frac{\partial^2}{\partial x_1^2} + x_2 \frac{\partial}{\partial x_2} + x_1^2
\]

then we can re-express the above differential equation as

\[
\mathcal{L} [\phi] = x_1^2 + x_2^2
\]

### 2.2. More on linear PDEs.

**Remark 1.5.** The condition that a PDE

\[
\mathcal{L} [\phi] = g (x_1, \ldots, x_n)
\]

be linear amounts to the following condition on the differential operator \( \mathcal{L} \)

\[
\mathcal{L} [c_1 \phi + c_2 \psi] = c_1 \mathcal{L} [\phi] + c_2 \mathcal{L} [\psi]
\]

for all constants \( c_1, c_2 \) and all differentiable functions \( \phi \) and \( \psi \).

Equivalently, a PDE \( \mathcal{L} [\phi] = g (x_1, \ldots, x_n) \) is linear if the operator \( \mathcal{L} \) acts a *linear transformation* on the vector space of differentiable functions on \( \mathbb{R}^n \).
Remark 1.6. Just as when linear differential equations expressed in terms of differential operators, it is
common to distinguish PDEs of the form
\[ L[ϕ] = 0 \]
from those of the form
\[ L[ϕ] = g(x) \neq 0 \]
The former are called **homogeneous PDEs** and the latter **inhomogeneous PDEs**.

By a **solution** of the PDE (1) in a region \( R \subset \mathbb{R}^n \), we mean an explicit function \( \Phi : \mathbb{R}^n \rightarrow \mathbb{R} \) such that
\[ F[x, \Phi, \partial_i \Phi, \partial_i \partial_j \Phi, \ldots, \partial_i \partial_j \ldots \partial_k \Phi](x) \]
vanishes identically at each point \( x \in R \). Note that if (1) has degree \( d \) then \( \Phi \) must be of class \( C^d \) (i.e., \( \Phi \) and each of partial derivatives up to order \( d \) must be continuous throughout \( R \)).

**Theorem 1.7** (Superposition Principle). If \( ψ_1 \) and \( ψ_2 \) are two solutions of a linear homogeneous PDE
\[ L[ϕ] = 0, \]
then any function of the form
\[ ψ = c_1 ψ_1 + c_2 ψ_2 \]
is also a solution of \( L[ϕ] = 0 \).

**Theorem 1.8.** If \( Ψ \) is a solution of a inhomogeneous linear PDE
\[ L[ϕ] = g \]
and \( ψ \) is any solution of the corresponding homogeneous linear PDE
\[ L[ϕ] = 0 \]
Then \( Ψ'(x) = Ψ(x) + ψ(x) \) is also a solution of the original inhomogeneous linear PDE.

### 3. Review of ODEs

Before undertaking our study of partial differential equations, let’s take a minute to review some of the
theory of ordinary differential equations.

Let’s begin with the simplest possible ODE:
(2) \[ \frac{dx}{dt} = 0 \]

because this already tells us something important. This equation says that the rate of change of \( x \) with
respect to \( t \) is 0. In other words,
\[ \frac{dx}{dt} = 0 \Rightarrow x(t) = \text{a constant}. \]

But the differential equation itself does not tell us what the constant is; indeed it can be any constant. One
customarily writes the solution to
\[ \frac{dx}{dt} = 0 \]
as
(3) \[ x(t) = c \]
and thinks of the constant \( c \) as a parameter that once fixed specifies a particular solution of \( \frac{dx}{dt} = 0 \). The
right hand side of (3) (with \( c \) interpreted as a variable parameter) is referred to as the **general solution** of
(2).

As a second simple example of an ODE consider
(4) \[ \frac{dx}{dt} = f(t) \]
where \( f(t) \) is some (continuous) function of \( t \). The Fundamental Theorem of Calculus tells us that

\[
x(t) - x(t_0) = \int_{t_0}^{t} \left( \frac{dx}{dt} \right) dt = \int_{t_0}^{t} f(t) dt = F(t) - F(t_0)
\]

where \( F(t) \) is any anti-derivative of \( f(t) \) (i.e. \( F(t) \) is any function whose derivative is \( f(t) \); which is, of course is calculated by integrating \( f(t) \) using various formulas for integration). If we set

\[
F(t) = \int f(t) dt
\]

and

\[
C = x(t_0) - F(t_0) \quad \text{(in toto some constant)}
\]

then we have

\[
x(t) = \int f(t) dt + C
\]

as the general solution of (4). As in the preceding example, our main point is that the general solution (from which all other solutions are a specialization) involves a single arbitrary constant.

More generally,

**Theorem 1.9.** (Existence and Uniqueness Theorem for 1st order ODEs). Suppose \( F: \mathbb{R}^2 \to \mathbb{R} \) is such that both \( F(x, t) \) and \( \frac{\partial F}{\partial x}(x, t) \) are differentiable functions of \( x \) and \( t \) around a point \((t_0, x_0) \in \mathbb{R}^2\). Then there exists one and only one solution to the ODE

\[
\frac{dx}{dt} = F(x, t)
\]

such that

\[
x(t_0) = x_0.
\]

This theorem gives not only a simple check for the existence of solutions (we can expect solutions that are valid around all points \((x, t)\) where \( F \) and \( \frac{\partial F}{\partial x} \) exist and are differentiable) but it also tells us the (the graphs of) solutions can only cross each other at singular points (where the hypothesis of the theorem does not hold). Fixing an initial value \( t_0 \) of \( t \), we can then parameterize the solutions valid around \( t = t_0 \) by the value of \( x \) at \( t_0 \).

\[
x_0 \in \mathbb{R} \iff \text{unique solution } \phi(t) \text{ of } \frac{dx}{dt} = F(x, t) \text{ with } \phi(t_0) = x_0
\]

Another way of putting this is that the general solution of a first order ODE should involve a single parameter. However, we quickly add that specifying the value of a solution at a particular point, is just one way of selecting a particular solution. More often, in practice, the parameters which appear in the general solution of differential equation arise as constants of integration that appear in the course of undoing derivatives (as in the solution (6) of (4)).

Even more generally, if

\[
\frac{d^n x}{dt^n} = F\left(t, x, \frac{dx}{dt}, \ldots, \frac{d^{n-1} x}{dt^{n-1}}\right)
\]

is an \( n^{th} \) order differential equation, its general solution (near generic points) will involve \( n \) arbitrary constants, because, loosely speaking, there will be \( n \) derivatives to undo, and so \( n \) arbitrary constants of integration in the solution. If the ODE (7) is in fact a linear ODE, then the solution space will, in fact, be an \( n \)-dimensional vector space.

**4. Linear ODEs**

Let me next recall some basic formulas for handling linear ordinary differential equations.
4. Linear ODEs

4.1. First order linear ODEs. The most general form of a first order linear ordinary differential equation is

\[ a(x) y' + b(x) y + c(x) = 0 \]

However, the formulas I give below for solutions of such an equation require one to first recast the such a first order linear ODE in \textit{standard form}

\[ y' + p(x) y = g(x) \]  

where

\[ p(x) = \frac{b(x)}{a(x)}, \quad g(x) = -\frac{c(x)}{a(x)} \]

We say such a differential equation is \textit{homogeneous} if \( g(x) = 0 \); i.e., the differential equation can be written

\[ y' + p(x) y = 0 \]

The general solution to (9) is

\[ y(x) = Cy_0(x) \]

where \( C \) is a constant and

\[ y_0(x) \equiv \exp \left( -\int p(x) \, dx \right) \]

Note that (9) is a homogenous linear equation and that (10) says that the solution space is 1-dimensional (consisting of the scalar multiples of a single “vector”).

Recall also the special solution (10) of (9) is also used to construct the the general solution (the more general situation) (8) via

\[ y' + p(x) y = g(x) \quad \Rightarrow \quad y(x) = y_0(x) \int \frac{g(x)}{y_0(x)} \, dx + Cy_0(x) \]

where \( y_0(x) \) is given by (10) and \( C \) is again a constant.

So in both case (equations (9) and (8)) we have infinitely many solutions (because the constant \( C \) can be any number). But because we have only one arbitrary number entering our formulas for the general solution, we can say, a little more precisely, that we have a 1-parameter family of solutions.

To get a unique solution (say one that corresponds to a particular experimental situation), one must place an additional condition on the solution; typically in the form of an \textit{initial condition}

\[ y(x_0) = y_0 \]

Such a condition furnishes one with an additional equation that can be used to solve for \( C \), thereby removing the arbitrariness in the general solution, and leaving one with a unique solution.

4.2. Second order linear ODEs. The standard form of a second order linear ODE

\[ a(x) y'' + b(x) y' + c(x) y + d(x) = 0 \]

is

\[ y'' + p(x) y' + q(x) y = g(x) \]

Like the case of first order linear ODEs the general solution of (11) is calculable from solutions of the corresponding homogeneous equation

\[ y'' + p(x) y' + q(x) y = 0 \]

Unfortunately, there is no closed formula analogous to (10) for solutions of (12). However, if one can find (or even guess) one solution of (12) the general solution to (11) can be calculated. This goes as follows.
Given one solution \( y_1(x) \) of (12) a second, independent solution of (12) can be calculated via the Reduction of Order formula

\[
y_2(x) = y_1(x) \int \frac{1}{(y_1(x))^2} \exp \left( -\int p(x) \right) dx
\]

In terms of \( y_1 \) and \( y_2 \) the general solution to (11) is then

\[
y(x) = -y_1 \int \frac{y_2 g}{y_1 y_2 - y'_1 y_2} dx + y_2 \int \frac{y_1 g}{y_1 y_2 - y'_1 y_2} dx + c_1 y_1 + c_2 y_2
\]

where \( c_1, c_2 \) are arbitrary constants. Thus we have a two parameter family of solutions to (11). To get a unique solution, one needs two additional conditions to fix the values of \( c_1 \) and \( c_2 \). This is typically done by fixing the value of \( y \) and its first derivative at a particular point:

\[
y(x_0) = y_0 \]
\[
y'(x_0) = y'_0
\]

5. First Order Linear PDEs

Here again the simplest case will tell us something fundamental. Consider

\[
\phi_x = 0 \quad \text{(i.e. } \frac{\partial \phi}{\partial x} = 0)\]

This says that the function \( \phi \) does not change with the \( x \) coordinate is varied. The general solution is thus¹

\[
\phi(x, y) = f(y) , \quad f \text{ an arbitrary function of the “other” variable } y
\]

Thus, in this simplest example we see already that the general solution to a PDE may involve arbitrary functions. We contrast this situation with the analogous ordinary differential equation (2) where general solution involved only a single arbitrary constant. In the present case, it will not suffice to specify that value of the solution at a point \((x_0, y_0)\) to get a unique solution; because such a condition will fix only the value of \( f(y) \) when \( y = y_0 \); it will not fix the values of \( f \) at other points \( y \), and it so will not determine the function \( f(y) \)

As a second simple example consider

\[
au_x + bu_y = 0
\]

where \( a, b \) are constants not both zero.

5.1. Geometric construction of solution. Let \( v \) be a vector in \( \mathbb{R}^n \). Recall that the directional derivative \( D_vf(x) \) of a differentiable function \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) at a point \( x \in \mathbb{R}^n \) is defined as

\[
D_vf(x) = \lim_{\varepsilon \to 0} \frac{f(x + \varepsilon v) - f(x)}{\varepsilon}
\]

and is computable as

\[
D_vf(x) = v \cdot \nabla f(x) := v_1 \frac{\partial f}{\partial x_1}(x) + v_2 \frac{\partial f}{\partial x_2}(x) + \cdots + v_n \frac{\partial f}{\partial x_n}(x)
\]

(i.e. \( D_vf(x) \) is the dot product of the gradient of \( f \) at \( x \) with the vector \( v \)).

The PDE

\[
au_x + bu_y = 0
\]

¹Actually, it is not clear from the way the PDE is written that there are only two underlying variables. For the expression \( \phi_x = 0 \) only indicates that \( x \) is an underlying variable of the function \( \phi \). In practice, the number of underlying variables will be clear from the context in which a PDE arises.
thus says that the function \( u(x, y) \) must be constant in the direction of \( v = (a, b) \). Put another way, any solution of (13) must be constant along any line parallel to the direction of \((a, b)\). Such lines are of the form

\[ \ell = \{(x, y) \in \mathbb{R}^2 \mid (x, y) = (c_1, c_2) + t(a, b)\} \]

from which we may infer

\[
\begin{align*}
x &= c_1 + ta \\
y &= c_2 + tb 
\end{align*}
\]

multiplying the first equation by \(-b\) and the second equation by \(a\) and then adding we obtain

\[-bx + ay = -bc_1 + ac_2 = \text{some constant}.\]

Thus the lines parallel to \((a, b)\) satisfy linear equations of the form

\[(14) \quad -bx + ay = \text{constant}\]

Since along such lines a solution of (8) is constant, it follows that the value of a solution \( u(x, y) \) is determined by which of these lines parallel to \( v \), the point \((x, y)\) sits on; which is in turn determined by the value of the constant on the right hand side of (14). Thus, we can infer that

\[ u(x, y) = f(-bx + ay) \]

for some arbitrary function \( f \).

### 5.2. Digression: Chain Rule for Functions of Several Variables.

In order to make a change of variables or to look at the behaviour of the solution of a PDE along a particular curve, it is important to understand how to implement the Chain Rule for functions of several variables. Let me take a minute to explain this formula in a fairly general setting.

Suppose you have a function \( \phi : \mathbb{R}^n \rightarrow \mathbb{R} \) of \( n \) variables \((x_1, \ldots, x_n)\) and a map

\[ f : \mathbb{R}^m \rightarrow \mathbb{R}^n : (y_1, \ldots, y_m) \mapsto (x_1(y_1, \ldots, y_m), x_2(y_1, \ldots, y_m), \ldots, x_n(y_1, \ldots, y_m)) \]

mapping the points of an \( m \)-dimensional space to the points in \( \mathbb{R}^n \). We can then form the composed function \( \Phi : \mathbb{R}^m \rightarrow \mathbb{R} : (y_1, \ldots, y_m) \mapsto \phi((x_1(y_1, \ldots, y_m), x_2(y_1, \ldots, y_m), \ldots, x_n(y_1, \ldots, y_m))) \)

The Chain Rule tells us how to compute the partial derivatives \( \frac{\partial \Phi}{\partial y_i} \)

\[
\frac{\partial \Phi}{\partial y_i} = \sum_{j=1}^{n} \frac{\partial x_j}{\partial y_i} \frac{\partial \phi}{\partial x_j}
\]

### 5.2.1. Application 1: Change of Variables.

Suppose we have a function \( \phi(x, y) \) and we make a change of variables; (for example, we might want to change from rectangular coordinates to polar coordinates). The Chain Rule tells us how to relate derivatives with respect to the old coordinates to derivatives with respect to the new coordinates. Suppose

\[
\begin{align*}
x &= r \cos(\theta) \\
y &= r \sin(\theta)
\end{align*}
\]

and we define

\[ \Phi(r, \theta) = \phi(r \cos(\theta), r \sin(\theta)) \]

Then the Chain rule says

\[
\begin{align*}
\frac{\partial \Phi}{\partial x} &= \frac{\partial \phi}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial \phi}{\partial y} \frac{\partial y}{\partial x} = \cos(\theta) \frac{\partial \phi}{\partial x} + \sin(\theta) \frac{\partial \phi}{\partial y} \\
\frac{\partial \Phi}{\partial \theta} &= \frac{\partial \phi}{\partial r} \frac{\partial r}{\partial \theta} + \frac{\partial \phi}{\partial y} \frac{\partial y}{\partial \theta} = -r \sin(\theta) \frac{\partial \phi}{\partial x} + r \cos(\theta) \frac{\partial \phi}{\partial y}
\end{align*}
\]

You’ll see another example of such a change of variables computation when we discuss the coordinate method for solving \( a \phi_x + b \phi_y = 0 \), just below.
5.2.2. Application 2: Derivative of a function along a curve. Suppose we want to know you the rate of change of a function \( \phi(x, y) \) changes as we move along a particular curve \( y = f(x) \). This could be computed by setting

\[
\Phi(x) = \phi(x, f(x))
\]

and then computing \( \frac{d\Phi}{dx} \). Here we regard \( \Phi(x) \) as the function obtained by composing \( g: \mathbb{R} \to \mathbb{R}^2: t \mapsto (t, f(t)) \) with \( \phi(x, y) \). The Chain Rule then tells us that

\[
\frac{d\Phi}{dt} = \frac{dx}{dt} \frac{\partial\phi}{\partial x} + \frac{dy}{dt} \frac{\partial\phi}{\partial y} = \frac{\partial\phi}{\partial x} + f(x) \frac{\partial\phi}{\partial y}
\]

We’ll see an explicit application of this formula in the next lecture.

5.3. Coordinate Method. Consider the vector \( \mathbf{v} := (b, -a) \). We have

\[
\mathbf{v} \cdot \mathbf{v}^\perp = ab - ba = 0
\]

so \( \mathbf{v}^\perp \) is perpendicular to \( \mathbf{v} \). The pair \( \mathbf{v}, \mathbf{v}^\perp \) thus constitute a pair of orthogonal directions in the plane. Now set

\[
\begin{align*}
  s &= ax + by \\
  t &= -bx + ay
\end{align*}
\]

with inverse relations

\[
\begin{align*}
  x &= \frac{1}{a^2 + b^2} (bs + at) \\
  y &= \frac{1}{a^2 + b^2} (as - bt)
\end{align*}
\]

regarding \( s, t \) as a new set coordinates. Using the chain rule for functions of two variables we can express derivatives with respect to \( x, y \) in terms of derivatives with respect to the new variables \( s \) and \( t \). Thus, if

\[
\tilde{\phi}(s, t) \equiv \phi(x(s, t), y(s, t))
\]

we will have

\[
\begin{align*}
  \frac{\partial\phi}{\partial x} &= \frac{\partial s}{\partial x} \frac{\partial\tilde{\phi}}{\partial s} + \frac{\partial t}{\partial x} \frac{\partial\tilde{\phi}}{\partial t} = a \frac{\partial\tilde{\phi}}{\partial s} - b \frac{\partial\tilde{\phi}}{\partial t} \\
  \frac{\partial\phi}{\partial y} &= \frac{\partial s}{\partial y} \frac{\partial\tilde{\phi}}{\partial s} + \frac{\partial t}{\partial y} \frac{\partial\tilde{\phi}}{\partial t} = b \frac{\partial\tilde{\phi}}{\partial s} + a \frac{\partial\tilde{\phi}}{\partial t}
\end{align*}
\]

So if

\[
0 = a\phi_x + b\phi_y
\]

we must have

\[
0 = a \left( a \frac{\partial\tilde{\phi}}{\partial s} - b \frac{\partial\tilde{\phi}}{\partial t} \right) + b \left( \frac{\partial\tilde{\phi}}{\partial s} + a \frac{\partial\tilde{\phi}}{\partial t} \right)
\]

\[
= a^2 \frac{\partial\tilde{\phi}}{\partial s} - ab \frac{\partial\tilde{\phi}}{\partial t} + b^2 \frac{\partial\tilde{\phi}}{\partial s} + ba \frac{\partial\tilde{\phi}}{\partial t}
\]

\[
= (a^2 + b^2) \frac{\partial\tilde{\phi}}{\partial s}
\]

Dividing the extreme sides of this chain of equalities by \( a^2 + b^2 \) we get

\[
0 = \frac{\partial\tilde{\phi}}{\partial s}
\]

So the function \( \tilde{\phi}(s, t) \) correspoding to a solution of \( a\phi_x + b\phi_y = 0 \) by a simple change of variables cannot depend on \( s \): Thus

\[
\tilde{\phi}(s, t) = f(t)
\]

But then

\[
\phi(x, y) = \tilde{\phi}(s(x, y), t(x, y)) = f(t(x, y)) = f(-bx + ay)
\]

which is the same answer we got before.
Example 1.10. Let's consider again the PDE

\begin{equation}
\alpha \phi_x + \beta \phi_y = 0
\end{equation}

but this time coupled with a boundary condition

\begin{equation}
\phi(0, y) = f(y)
\end{equation}

We'll solve this PDE with boundary condition yet another way.

Consider the curves (actually a line) $\gamma_c : t \mapsto [x(t), y(t)] \in \mathbb{R}^2$ where

\begin{align*}
x(t) &= at \\
y(t) &= bt + c
\end{align*}

the curve $\gamma_c$ is just a straight line with tangent vector

\begin{align*}
\frac{d\gamma_c}{dt} &= \left[ \frac{dx}{dt}, \frac{dy}{dt} \right] = [a, b]
\end{align*}

Now suppose $\phi(x, y)$ is a solution of $\alpha \phi_x + \beta \phi_y = 0$. How does the solution $\phi$ vary as we move along the curve $\gamma_c$? Well, by the Chain Rule,

\begin{align*}
\frac{d}{dt} (\phi(\gamma_c(t))) &= \frac{\partial \phi}{\partial x} \frac{dx}{dt} + \frac{\partial \phi}{\partial y} \frac{dy}{dt} = \alpha \phi_x + \beta \phi_y = 0
\end{align*}

In other words, a solution $\phi$ must remain constant as one moves along a curve $\gamma_c(t)$.

But what is the actual value of the solution at an arbitrary point $(x, y)$? Well, the point $(x, y)$ sits on one of the curves $\gamma_c(t)$ for some choice of $c$ and $t$. Indeed, if we set

\begin{align*}
x &= at \\
y &= bt + c
\end{align*}

we can solve this pair of equations for $t$ and $c$. One gets

\begin{align*}
t &= \frac{x}{a} \\
c &= y - \frac{b}{a} x
\end{align*}

Thus, the point $(x, y)$ lies on the curve $\gamma_c$ where $c = y - \frac{b}{a} x$. But the point $(0, c) = (0, y - \frac{b}{a} x)$ also lies on this same curve. So the value of the solution at $(x, y)$ must be the same as its value at $(0, y - \frac{b}{a} x)$. Thus,

\begin{align*}
\phi(x, y) &= \phi\left(0, y - \frac{b}{a} x\right) = f\left(y - \frac{b}{a} x\right)
\end{align*}

Noting that there was nothing special about the way we chose the point $(x, y)$ we can infer the value of the solution $\phi$ to (15) - (16) any point $(x, y)$ is given by

\begin{align*}
\phi(x, y) &= f\left(y - \frac{b}{a} x\right)
\end{align*}

Example 1.11. Find the solution of

\begin{align*}
\phi_x &= x \\
\phi(0, y) &= y^2
\end{align*}

Here we’ve introduced two complications to our initial example: we’ve introduced an inhomogeneous term to the right hand side of the PDE and we’ve imposed a certain boundary condition along the $y$-axis.
Now PDE is again just telling us how a solution must change along the $x$-direction; it’s just that now it’s changing non-trivially in the $x$-direction. In fact, suppose we set $\Phi_{y_0}$ to be the restriction of the solution to the line $y = y_0$. This we can regard as a function of $x$ alone

$$\Phi_{y_0}(x) = \phi(x, y_0)$$

And we have, by the chain rule

$$\frac{d}{dx} \Phi_{y_0}(x) = \frac{\partial \phi}{\partial x} \frac{dx}{dx} + \frac{\partial \phi}{\partial y} \frac{dy_0}{dx} = \frac{\partial \phi}{\partial x}$$

(because $\frac{dx}{dx} = 1$ and $\frac{dy_0}{dx} = 0$). The original PDE says that the right-hand side must be $x$. Hence, $\Phi_{y_0}(x)$ must satisfy

$$\frac{d\Phi_{y_0}}{dx} = x \implies \Phi_{y_0}(x) = \frac{1}{2}x^2 + C_{y_0}$$

Thus, we know

$$\phi(x, y_0) = \Phi_{y_0}(x) = \frac{1}{2}x^2 + C_{y_0}$$

i.e., we know the solution at all points $(x, y_0)$ up to a term that depends only on $y_0$. We may as well write this as

$$\phi(x, y) = \frac{1}{2}x^2 + C(y)$$

To figure out $C(y)$ we just need to impose the boundary condition.

$$y^2 = \phi(0, y) = \frac{1}{2}(0)^2 + C(y) \implies C(y) = y^2$$

And so we have

$$\phi(x, y) = \frac{1}{2}x^2 + y^2$$

The moral of the preceding example is that we can sometimes solve a first order linear PDE by interpreting it as telling us how the solution must change along a particular direction. In the next lecture, we’ll generalize this idea and solve first order linear PDEs by interpreting them as telling us how solutions must vary along a particular curves.