1. Show that a function \( f(z) = u(z) + iv(z) \) of a complex variable \( z = x + iy \) that satisfies the Cauchy-Riemann equations
\[
\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}
\]
also has the property that both its real part \( u(z) \) and its imaginary part \( v(z) \) satisfy Laplace’s equation: i.e.,
\[
u_{xx} + u_{yy} = 0 = v_{xx} + v_{yy}
\]
2. Let \( g(x) \) be any piecewise continuous function on \( \mathbb{R} \). Show directly from the definition, that the mapping \( \phi_g : C_c^\infty \to \mathbb{R} \) given by
\[
\phi_g(f) := \int_{-\infty}^{\infty} f(x) g(x) \, dx
\]
defines a distribution. It will be easy to show that \( \phi_g \) defines a linear functional. The hard part will be to demonstrate that \( \phi_g \) is continuous. To this end, show that if \( \{f_n\}_{n \in \mathbb{N}} \subset C_c^\infty(\mathbb{R}) \) converges uniformly to a function \( f(x) \in C_c^\infty(\mathbb{R}) \), then
\[
\lim_{n \to \infty} \phi_g(f_n) = \phi_g(f)
\]
By the way, uniform convergence means the following
- \( \{f_n\} \) converges uniformly to \( f \) if for every \( \varepsilon > 0 \) there exists a natural number such that \( |f_n(x) - f(x)| < \varepsilon \) for all \( x \in \mathbb{R} \) and all \( n > N \).
3. Let \( \psi \) be any distribution. Show that the functional \( \psi' \) defined by
\[
\psi'(f) := \psi \left( \frac{df}{dx} \right)
\]
is a distribution.
4. Let \( u(x) = u(x,y) \) be a solution of Laplace’s equation \( \nabla^2 u(x) = 0 \) on a planar domain \( D \).
   (a) Show that the function
\[
f(x) = \ln ||x||
\]
is a solution of Laplace’s equation on \( \mathbb{R}^2 \) except at \( x = 0 \). (Hint: change to polar coordinates.)
   (b) Let \( f_\varepsilon(x) = \ln (||x|| + \varepsilon) \). Show that
   - (i) \( \lim_{\varepsilon \to 0} \nabla^2 f_\varepsilon(x) = 0 \) whenever \( x \neq 0 \).
   - (ii) \( \int_{\mathbb{R}^2} \nabla^2 f_\varepsilon(x) \, dA = 2\pi \) independent of \( \varepsilon \)
   and conclude that
   \[
   \lim_{\varepsilon \to 0} \frac{1}{2\pi} \nabla^2 \ln (||x|| + \varepsilon) = \delta^{(2)}(x)
   \]
   (the 2-dimensional delta functional).
   (c) Use Green’s Identity
\[
\int_D \phi \nabla^2 \psi \, dA = \int_D \psi \nabla^2 \phi \, dA + \int_{\partial D} (\phi \nabla \psi - \psi \nabla \phi) \cdot \mathbf{n} \, ds
\]
and the results of (b) to derive the representation formula
\[
u(x_0) = \frac{1}{2\pi} \int_{\partial D} [u(x)(\nabla \ln ||x - x_0||) - (\nabla u(x)) \ln ||x - x_0||] \cdot \mathbf{n} \, ds
\]
that expresses a solution \( u \) of Laplace’s equation at an interior point \( x_0 \in D \) as a certain integral of \( u \) and its gradient over the boundary of \( D \).