1. Use the Maximum Principle for the Heat Equation to demonstrate that there is a unique solution to
\[
\begin{align*}
    u_t - k^2 u_{xx} &= f(x,t), \quad 0 \leq x \leq L, \quad t > 0 \\
    u(0,t) &= g(t), \quad t > 0 \\
    u(L,t) &= h(t), \quad t > 0 \\
    u(x,0) &= \phi(x), \quad 0 \leq x \leq L
\end{align*}
\] (1a–1d)

2. Prove the following identities
\[
\int_{-\pi}^{\pi} \sin(mx) \sin(nx) \, dx = \begin{cases} 
\pi & \text{if } m = n \\
0 & \text{if } m \neq n 
\end{cases} \quad (2a)
\]
\[
\int_{-\pi}^{\pi} \sin(mx) \cos(nx) \, dx = 0 \quad (2b)
\]
\[
\int_{-\pi}^{\pi} \cos(mx) \cos(nx) \, dx = \begin{cases} 
\pi & \text{if } m = n \\
0 & \text{if } m \neq n 
\end{cases} \quad (2c)
\]

3. Consider the following Heat Equation boundary value problem:
\[
\begin{align*}
    u_t - k^2 u_{xx} &= 0, \quad 0 \leq x \leq L, \quad t > 0 \\
    u(0,t) &= 0, \quad t > 0 \\
    u(L,t) &= 0, \quad t > 0 \\
    u(x,0) &= \phi(x), \quad 0 \leq x \leq L
\end{align*}
\] (3a–3d)

(a) Apply the method of Separation of Variables to find a family of solutions of (3a) in the form \( u(x,t) = X(x)T(t) \).

(b) Impose the boundary conditions (3b) and (3c) to find a more specialized family of solutions \( u_n(x,t) = X_n(x)T_n(t) \) satisfying (1a)–(1c).

(c) Set
\[
u(x,t) = \sum_n a_n u_n(x,t)
\]
where the \( u_n(x,t) \) are the solutions found in (b), impose (3d), and then use properties of Fourier expansions to determine the coefficients \( a_n \).

4. Find the solution of the following PDE/BVP:
\[
\begin{align*}
    u_t - u_{xx} &= 0, \quad 0 \leq x \leq 1, \quad t > 0 \\
    u(0,t) &= 0, \quad t > 0 \\
    u(1,t) &= 0, \quad t > 0 \\
    u(x,0) &= 1 - x^2, \quad 0 \leq x \leq 1
\end{align*}
\] (4a–4d)