Math 4233

Homework Set 6

Problem 1. Find the first three non-zero terms in each of two linearly independent solutions of

$$xy'' + y' - y = 0$$

valid near $x = 0$.

- The differential equation has a regular singular point at $x = 0$, so we'll apply the Method of Frobenius to get at least one solution as a generalized power series.

  Substituting $y = \sum_{n=0}^{\infty} a_n x^{n+r}$ into the differential equation we get

$$\begin{aligned} 0 &= x\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} + \sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} - \sum_{n=0}^{\infty} a_n x^{n+r} \\ &= \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-1} + \sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} + \sum_{n=0}^{\infty} -a_n x^{n+r} \end{aligned}$$

  To prepare to add these generalized power series expressions, we'll shift summation indices and peel off initial terms

$$\begin{aligned} \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-1} &= \sum_{n=-1}^{\infty}(n+r+1)(n+r)a_{n+1}x^{n+r} \\ &= (r)(r-1)a_0 x^{r-1} + \sum_{n=0}^{\infty}(n+r+1)(n+r)a_{n+1}x^{n+r} \end{aligned}$$

$$\begin{aligned} \sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} &= \sum_{n=-1}^{\infty}(n+r+1)a_{n+1}x^{n+r} \\ &= ra_0 x^{r-1} + \sum_{n=0}(n+r+1)a_{n+1}x^{n+r} \end{aligned}$$

  We now replace the expressions on the left as the appear in the power series expression of the differential equation, with the corresponding power expressions on the right:

$$\begin{aligned} 0 = (r)(r-1)a_0 x^{r-1} &+ \sum_{n=0}^{\infty}(n+r+1)(n+r)a_{n+1}x^{n+r} \\ &+ ra_0 x^{r-1} + \sum_{n=0}(n+r+1)a_{n+1}x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r} \end{aligned}$$

  or

$$\begin{aligned} 0 &= (r(r-1)+r)a_0 x^{r-1} + \sum_{n=0}^{\infty}\left[(n+r+1)(n+1)a_{n+1} + (n+r+1)a_{n+1} - a_n\right]x^{n+r+1} \\ &= r^2 a_o x^{r-1} + \sum_{n=0}^{\infty}\left[(n+r+1)^2 a_{n+1} - a_n\right]x^{n+r} \end{aligned}$$

  Setting the total coefficient of x^{r-1} (the lowest order term on the right) equal to 0 yields the **indicial equation**:

$$r^2 = 0 \quad \Rightarrow \quad r = 0$$

  (as by hypothesis $a_0 \neq 0$). Setting the total coefficient of x^{n+r} equal to zero ($n = 0, 1, 2, 3, \ldots$) yields the **recursion relations:**

$$a_{n+1} = \frac{a_n}{(n+r+1)^2} = \frac{a_n}{(n+1)^2} \quad .$$

where we have used the solution $r = 0$ of the indicial equation.

- To get a first solution we can now set $a_0 = 1$ and begin solving the recursion relations

$$
\begin{aligned}
a_1 &= \frac{a_0}{(0+1)^2} = 1 \\
a_2 &= \frac{a_1}{(1+1)^2} = \frac{1}{4} = \frac{1}{2^2} \\
a_3 &= \frac{a_2}{(2+1)^2} = \frac{1}{4}\frac{1}{9} = \frac{1}{(2\cdot 3)^2} \\
a_4 &= \frac{a_3}{(3+1)^2} = \frac{1}{(2\cdot 3\cdot 4)^2}
\end{aligned}
$$

In general one finds

$$
a_n = \frac{1}{(n!)^2}
$$

Thus,

$$
y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} = \sum_{n=0}^{\infty} \frac{1}{(n!)^2} x^n
$$

is our first solution.

- Since we only have one root of the indicial equation, the second solution will be of the form

$$
\begin{aligned}
y_2(x) &= y_1(x)\ln|x| + \sum_{n=1}^{\infty} b_n x^{n+r} \\
&= y_1(x)\ln|x| + \sum_{n=1}^{\infty} b_n x^n
\end{aligned}
$$

We'll plug the latter expression back into the differential equation to figure out the appropriate choice for the coefficients b_n. To ease some of the computations let us write

$$
b(x) = \sum_{n=1}^{\infty} b_n x^n
$$

We then have

$$
\begin{aligned}
y_2(x) &= y_1(x)\ln|x| + b(x) \\
y_2'(s) &= y_1'(x)\ln|x| + \frac{1}{x}y_1(x) + b'(x) \\
y_2''(x) &= y_1''(x)\ln|x| + \frac{2}{x}y_1'(x) - \frac{1}{x^2}y_1(x) + b''(x)
\end{aligned}
$$

So our condition on the coefficients b_n is that

$$
\begin{aligned}
0 &= xy_2'' + y_2' - y_2 \\
&= (xy_1'' + y_1' - y_1)\ln|x| + 2y_1' - \frac{1}{x}y_1 + xb'' + \frac{1}{x}y_1 + b' - b \\
&= 0 + 2y_1' + xb'' + b' - b
\end{aligned}
$$

since y_1, by construction, satisfies $xy'' + y' - y = 0$ and the $\pm\frac{1}{x}y_1$ terms cancel. Thus,

$$\begin{aligned}
0 &= 2y' + b'' + b' - b \\
&= \sum_{n=0}^{\infty} 2na_n x^{n-1} + \sum_{n=1}^{\infty} n\left(n-1\right) b_n x^{n-1} + \sum_{n=1}^{\infty} nb_n x^{n-1} - \sum_{n=1}^{\infty} b_n x^n \\
&= \sum_{n=-1}^{\infty} 2\left(n+1\right) a_{n+1} x^n + \sum_{n=-1}^{\infty} \left(n+1\right)\left(n\right) b_{n+1} x^n + \sum_{n=0}^{\infty} \left(n+1\right) b_{n+1} x^n - \sum_{n=1}^{\infty} b_n x^n \\
&= 0 + 2a_1 + \sum_{n=1}^{\infty} 2\left(n+1\right) a_{n+1} x^n + 0 + \sum_{n=2}^{\infty} n\left(n+1\right) b_{n+1} x^n \\
&+ b_1 + \sum_{n=1}^{\infty} \left(n+1\right) b_{n+1} x^n - \sum_{n=1}^{\infty} b_n x^n \\
&= \left(2a_1 + b_1\right) + \sum_{n=1}^{\infty} \left[2\left(n+1\right) a_{n+1} + n\left(n+1\right) b_{n+1} + \left(n+1\right) b_{n+1} - b_n\right] x^{n-1} \\
&= \left(2a_1 + b_1\right) + \sum_{n=1}^{\infty} \left[2\left(n+1\right) a_{n+1} + \left(n+1\right)^2 b_{n+1} - b_n\right] x^{n-1}
\end{aligned}$$

Setting the total coefficient of each distinct power of x equal to 0 yields

$$\begin{aligned}
b_1 &= -2a_1 \\
b_{n+1} &= \frac{b_n - 2\left(n+1\right) a_n}{\left(n+1\right)^2} \qquad , \qquad n = 1, 2, 3, \ldots
\end{aligned}$$

Thus, now substituting in our known formulas for a_n,

$$\begin{aligned}
b_1 &= -2a_1 = -2 \\
b_{n+1} &= \frac{b_n}{\left(n+1\right)^2} - \frac{2}{\left(n+1\right)} \frac{1}{\left(n!\right)^2}
\end{aligned}$$

We have thus have

$$b\left(x\right) = -2x - \frac{3}{4}x^2 - \frac{11}{108}x^3$$

and so

$$y_2 = \left(1 + x + \frac{1}{4}x^2 + \frac{1}{36}x^3 + \cdots\right) \ln|x| + \left(-2x - \frac{3}{4}x^2 - \frac{11}{108}x^3\right)$$

Problem 2. A vibrating drum head obeys the following PDE

$$\frac{\partial^2 \phi}{\partial t^2} - c^2 \nabla^2 \phi = 0 \tag{1}$$

where ∇^2 is the 2-dimensional Laplacian

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

in polar coordinates. In this problem, you'll use separation of variables to formulate a (fairly general) solution and then solve a particular boundary value problem.

(a) Use Separation of Variables to reduce the PDE to a set of three weakly coupled ODEs.

- Substituting $\phi(r,\theta,t) = c^2 R(r)\,\Theta(\theta)\,T(t)$ into the PDE (1) we get

$$R\Theta T'' - c^2 R''\Theta T - c^2 \frac{1}{r} R'\Theta T - c^2 \frac{1}{r^2} R\Theta'' T = 0$$

or, after dividing by $R\Theta T$ and moving the r and θ dependent terms to the right hand side,

$$\frac{1}{c^2}\frac{T''}{T} = \left(\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta}\right)$$

Because the left hand side depends only on t while the right hand side depends only on r and θ, both sides must equal a constant. Let's call it $-\lambda^2$. We then have

$$\frac{T''}{T} = -c^2\lambda^2 \quad \Rightarrow \quad T'' + c^2\lambda^2 T = 0$$

and

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} = -\lambda^2$$

or

$$r^2\frac{R''}{R} + r\frac{R'}{R} + \lambda^2 r^2 = -\frac{\Theta''}{\Theta}$$

Again the usual Separation of Variables argument can be applied to deduce a pair of ODEs coupled weakly by a separation constant that we'll denote by α^2 :

$$-\frac{\Theta''}{\Theta} = \alpha^2 \quad \Rightarrow \quad \Theta'' + \alpha^2\Theta = 0$$

and

$$r^2\frac{R''}{R} + r\frac{R'}{R} + \lambda^2 r^2 = \alpha^2 \quad \Rightarrow \quad r^2 R'' + rR' + \left(\lambda^2 r^2 - \alpha^2\right) R = 0$$

- Our system of three weakly coupled ODEs is thus

$$\begin{aligned} T'' + c^2\lambda^2 T &= 0 \\ \Theta'' + \alpha^2\Theta &= 0 \\ r^2 R'' + rR' + \left(\lambda^2 r^2 - \alpha^2\right) R &= 0 \end{aligned}$$

(b) Use periodicity with respect to the angular variable θ to put a restriction on one of the separation constants.

- Since θ corresponds to the angle in the xy-plane, we must have $\phi(r,\theta,t) = \phi(r,\theta+2\pi,t)$ and this in turn requires the separation parameter α to be a integer (in fact, without loss of generality, a non-negative integer). Setting $\alpha = n \in \mathbb{N}$, we then have

$$\Theta(\theta) = a_n\cos(n\theta) + b_n\sin(n\theta)$$

(c) Note that the radial factors $R_{n,\lambda}(r)$ of the Separation of Variables solutions have a regular singularity at $r = 0$. Make a change of variables $r \to \lambda x$ to recast the differential equation as a differential equation of the form

$$x^2y'' + xy' + (x^2 - n^2)y = 0$$

This differential equation is called *the Bessel equation of order* n and its solutions are usually denoted by $J_n(x)$.

- The radial equation (at the end of part (a)) is

  $$r^2R'' + rR' + \left(\lambda^2r^2 - n^2\right)R = 0$$

  Let $x = \lambda r$ and set

  $$y(x) = R(r(x)) = R\left(\frac{x}{\lambda}\right) \quad \Longleftrightarrow \quad R(r) = y(x(r)) = y(\lambda r)$$

  Then, by the chain rule

  $$y'(x) = \frac{1}{\lambda}R'\left(\frac{x}{\lambda}\right) \quad \Rightarrow \quad R' = \lambda y'$$
  $$y''(x) = \frac{1}{\lambda^2}R''\left(\frac{x}{\lambda}\right) \quad \Rightarrow \quad R'' = \lambda^2 y''$$

  and so

  $$\begin{aligned} 0 &= r^2R''(r) + rR'(r) + \left(\lambda^2 - n^2\right)R(r) \\ &\Rightarrow \quad r^2\left(\lambda^2y''(x)\right) + r(\lambda y'(x)) + \left(\lambda^2r^2 - n^2\right)y(x) = 0 \\ &\Rightarrow \quad x^2y''(x) + xy'(x) + \left(x^2 - n^2\right)y(x) = 0 \end{aligned}$$

- I note, at this point, that although we seem to have gotten rid of the separation parameter λ, we must not forget about it. It will resurface later in the problem when we'll need it not only to establish needed boundary conditions, but also because our functions $T_\lambda(t)$ depend on it.

(d). Show that the corresponding radial solutions $R_{n,\lambda}(r) = J_n(\lambda r)$ satisfy the orthogonality conditions

$$(3) \qquad \int_0^b R_{n,\lambda}(r)\,R_{n,\lambda'}(r)\,rdr = 0 \qquad \text{if } \lambda \neq \lambda'$$

provided the functions $R_{n,\lambda}(r)$ satisfy a boundary condition of the form

$$\alpha R_{n,\lambda}(b) + \beta R_{n,\lambda}(b) = 0$$

(This is very similar to the Sturm-Liouville situation, but because we have only one boundary condition, there is an additional subtlety.)

- We first note that the radial equation

  $$r^2R'' + rR' + \lambda^2r^2R - n^2R = 0$$

  after dividing by r can be written

  $$rR'' + R' - \frac{n^2}{r}R = -\lambda^2 rR$$

  or

  $$\frac{d}{dr}\left(r\frac{dR}{dr}\right) - \frac{n^2}{r}R = -\lambda^2 rR$$

  which has the form of a Sturm-Liouville ODE

  $$\frac{d}{dr}\left(\widetilde{p}(r)\frac{dR}{dr}\right) + \widetilde{q}(r)R = -\lambda^2\widetilde{r}(r)R$$

with

$$\widetilde{p}(r) = r \qquad , \qquad \widetilde{q}(r) = -\frac{1}{r} \qquad \text{and} \qquad \widetilde{r}(r) = r$$

(Sorry for the confusing notation, the function $\widetilde{p}(r)$, $\widetilde{q}(r)$ and $\widetilde{r}(r)$ are the analogs of the functions $p(x)$, $q(x)$ and $r(x)$ in Lecture 15; it's just that here we're already using r to denote the radial coordinate.) What I aim to do here is derive an analog of Lagrange's identity (Theorem 15.4) and from that the orthogonality relation.

We have (essentially repeating the double integration by parts calculation appearing in the proof of Theorem 15.4),

$$\begin{aligned}\int_0^b \phi \frac{d}{dr}\left(r\frac{d\psi}{dr}\right) dr &= \phi\left(r\frac{d\psi}{dr}\right)\Big|_0^b - \int_0^b \frac{d\phi}{dr}\left(r\frac{d\psi}{dr}\right) dr \\ &= \phi\left(r\frac{d\psi}{dr}\right)\Big|_0^b - \psi r\frac{d\phi}{dr}\Big|_0^b + \int_0^b \psi\left(\frac{d}{dr}r\frac{d\phi}{dr}\right) dr\end{aligned}$$

Thus, if we set

$$L[\psi] = \frac{d}{dr}\left(r\frac{d\psi}{dr}\right) - \frac{n^2}{r}\psi$$

we have

$$\begin{aligned}\int_0^b \phi L[\psi]\,\phi dr &= \int_0^b \phi\left(\frac{d}{dr}\left(r\frac{d\psi}{dr}\right) - \frac{n^2}{r}\psi\right) dr \\ &= \int_0^b \phi\left(\frac{d}{dr}\left(r\frac{d\psi}{dr}\right)\right) dr - \int_0^b \frac{n^2}{r}\phi\psi dr \\ &= \int_0^b \psi\left(\frac{d}{dr}r\frac{d\phi}{dr}\right) dr - \int_0^b \frac{n^2}{r}\phi\psi dr + \phi\left(r\frac{d\psi}{dr}\right)\Big|_0^b - \psi r\frac{d\phi}{dr}\Big|_0^b \\ &= \int_0^b \psi L[\phi]\, dr + \phi\left(r\frac{d\psi}{dr}\right)\Big|_0^b - \psi r\frac{d\phi}{dr}\Big|_0^b\end{aligned}$$

Thus,

$$\begin{aligned}\int_0^b (\phi L[\psi] - \psi L[\phi]) &= \left(\phi\left(r\frac{d\psi}{dr}\right) - \psi r\frac{d\phi}{dr}\right)\Big|_0^b \\ &= b\phi(b)\,\psi'(b) - b\psi(b)\,\phi'(b)\end{aligned}$$

because $r = 0$ at the lower endpoint of integration. This formula holds for any (differentiable) functions ϕ and ψ (we only used integration by parts and the form of L). Now suppose ϕ and ψ obey the following sort of boundary condition at $r = b$.

$$\alpha\phi(b) + \beta\phi'(b) = 0$$

If $\beta \neq 0$ then this condition implies

$$\begin{aligned}\phi'(b) &= -\frac{\alpha}{\beta}\phi'(b) \\ \psi(b) &= -\frac{\alpha}{\beta}\psi'(b)\end{aligned}$$

and so

$$b\phi(b)\,\psi'(b) - b\psi(b)\,\phi'(b) = b\phi(b)\left(\frac{-\alpha}{\beta}\psi(b)\right) - b\psi(b)\left(-\frac{\alpha}{\beta}\phi(b)\right) = 0.$$

And if $\beta = 0$, the boundary condition implies $\phi(b) = \psi(b) = 0$ and so

$$b\phi(b)\,\psi'(b) - b\psi(b)\,\phi'(b) = 0$$

even more directly. We can thus conclude:

– If ϕ, ψ are functions of r satisfying a boundary condition of the form

(*) $$\alpha\phi(b) + \beta\phi'(b) = 0$$

then

$$\int_0^b (\phi L[\psi] - \psi L[\phi])\, dr = 0 \quad .$$

- Let us now suppose further that the functions $\phi(r) = R_{n,\lambda_1}(r)$ and $\psi(r) = R_{n,\lambda_2}(r)$ are solutions of the radial differential equation

$$L[R_{\lambda_i}] = -r\lambda_i^2 R \qquad ; \qquad i = 1, 2$$

and also satisfy the boundary condition (*). Then

$$0 = \int_0^b (\phi L[\psi] - \psi L[\phi])\, dr = \int_0^b \left(\int_0^b R_{\lambda_1}\left(-\lambda_2^2 r R_{\lambda_2}\right) - R_{\lambda_2}\left(-\lambda_1^2 r R_{\lambda_1}\right) \right) dr$$
$$= \left(\lambda_1^2 - \lambda_2^2\right) \int_0^b r R_{\lambda_1}(r) R_{\lambda_2}(r)\, dr$$

and so

$$\lambda_1 \neq \lambda_2 \quad \Rightarrow \quad \int_0^b r R_{\lambda_1}(r) R_{\lambda_2}(r)\, dr = 0$$

(e) Use the Method of Frobenius (the generalized power series technique) to find the indicial equations and recursion relations for solutions of (2).

- Let's now solve the differential equation

$$x^2 y'' + x y' + \left(x^2 - n^2\right) y = 0$$

Of course, we'll be interested in solutions that are well-behaved at $x = 0$ (which corresponds to $r = 0$ in the original radial problem). But $x = 0$ is a regular singular point for this differential equation; therefore we'll have to resort to generalized power series and the Method of Frobenius.

Setting

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

we have

$$x^2 y'' = \sum_{k=0}^{\infty} (k+r)(k+r-1) a_k x^{k+r} = r(r-1) a_0 x^r + (r+1)(r) a_1 x^{r+1}$$
$$x y' = \sum_{k=0}^{\infty} (k+r) a_k x^{k+r} = r a_0 x^r + (r+1) a_1 x^{r+1} + \sum_{k=2}^{\infty} (k+r) a_k x^{k+r}$$
$$-n^2 y = \sum_{k=0}^{\infty} -n^2 a_k x^{k+r} = -n^2 a_0 x^r - n^2 a_1 x^{r+1} + \sum_{k=2}^{\infty} -n^2 a_k x^{k+r}$$
$$x^2 y = \sum_{k=0}^{\infty} a_k x^{k+r+2} = \sum_{k=2}^{\infty} a_{k-2} x^{k+r}$$

Thus, we need

$$0 = \left(r(r-1) + r - n^2\right) a_0 x^r + \left(r(r+1) + (r+1) - n^2\right) a_1 x^{r+1}$$
$$+ \sum_{k=2}^{\infty} \left[\left((k+r)(k+r-1) + (k+r) - n^2\right) a_k + a_{k-2}\right] x^{n+r}$$

So the indicial equation is

$$r^2 - n^2 = 0 \quad \Rightarrow \quad r = \pm n$$

Assuming the indicial equation holds, we also have (by demanding the total coefficient of x^{r+1} is 0)

$$\left((r+1)^2 - n^2\right) a_1 = 0 \quad \Rightarrow \quad a_1 = 0$$

and the recursion relations are

$$a_k = -\frac{a_{k-2}}{(k+r)^2 - n^2} \quad , \quad k = 2, 3, 4, \dots$$

(f) Find an explicit formulas for the functions $R_{n,\lambda}(r)$, $\Theta_n(\theta)$ and $T_\lambda(t)$ corresponding to Separation of Variables solutions $\phi_{n,\lambda}(r,\theta,t) = R_{n,\lambda}(r)\,\Theta_n(\theta)\,T_\lambda(t)$ to (1) that are radially symmetric (i.e., θ-independent) and regular at $r = 0$.

- We have

$$T''_\lambda + c^2\lambda^2 T = 0 \quad \Rightarrow \quad T_\lambda(t) = \alpha_\lambda \cos(\lambda c t) + \beta_\lambda \sin(\lambda c t)$$

and

$$\Theta''_n + n^2\Theta_n = 0 \quad \Rightarrow \quad \Theta_n(\theta) = \begin{cases} a_n \cos(n\theta) + b_n \sin(n\theta) & n \neq 0 \\ a_0 + b_0\theta & n = 0 \end{cases}$$

The solutions $\Theta_n(\theta)$ will be independent of θ only if $n = 0$ and $b_0 = 0$.

Let's now consider the differential equation for the radial function $R_{n,\lambda}(r)$ in the case where $n = 0$

$$r^2 R''_{0,\lambda} + r R'_{0,\lambda} + \lambda^2 r^2 R_{0,\lambda} = 0$$

(This is the only case we need for a radially symmetric solution of (1).) As in part (c), the solutions of this equation that are regular at $r = 0$ can be expressed as

$$R_{0,\lambda}(r) = J_0(\lambda r)$$

where $J_0(x)$ is a solution of

$$x^2 y'' + x y' + x^2 y = 0$$

that is regular at $x = 0$. Specializing the indicial equation and recursion relations found in part (e) to the case where $n = 0$, we have

$$r^2 = 0 \quad \Rightarrow \quad r = 0 \qquad \text{(indicial equation)}$$

$$a_1 = 0 \qquad \text{(condition on the coefficient of } x^{r+1}\text{)}$$

and

$$a_k = -\frac{a_{k-2}}{(k+r)^2 - n^2} = -\frac{a_{k-2}}{k^2} \quad , \quad k = 2, 3, 4, \dots$$

Because we have only one root of the indicial equation, we'll have one solution of the form

$$y_1(x) = \sum_{k=0}^{\infty} a_n x^{n+r}$$

and a second solution of the form

$$y_2(x) = y_1(x) \ln|x| + \sum_{k=1}^{\infty} b_k x^{k+r}$$

However, the second solution will not be regular at $r = 0$ (the log factor in the first term diverges as $r \to 0$). So we just need to write down explicit formulas for the coefficients a_n of $y_1(x)$. From the recursion relations it is clear that if $k = 2j$ is even

$$\begin{aligned} a_{2j} &= -\frac{a_{2j-2}}{(2j)^2} \\ &= +\frac{1}{(2j)^2}\frac{a_{2j-4}}{(2j-2)^2} \\ &\vdots \\ &= \frac{(-1)}{(2j)^2}\frac{(-1)}{(2j-2)^2}\cdots\frac{(-1)}{(2)^2}a_0 \\ &= \frac{(-1)^j}{2^{2j}(j!)^2}a_0 \end{aligned}$$

while if $k = 2j+1$ is odd

$$a_{2j+1} = \frac{(-1)}{(2j+1)^2}\cdots\frac{(-1)}{(1)^2}a_1 = 0$$

since $a_1 = 0$.

$$a_k = \begin{cases} (-1)^{k/2}\frac{a_0}{2^k(k!)^2} & \text{if } k \text{ is even} \\ 0 & \text{if } k \text{ is odd} \end{cases}$$

Thus,

$$R_{0,\lambda}(r) = y_1(\lambda r) = \sum_{j=0}^{\infty}\frac{(-1)^j a_0}{(2)^{2j}(2j!)^2}(\lambda r)^{2j}$$

(g). Write down a formula for the solution of (1) satisfying

$$\begin{aligned} \phi(b,\theta,t) &= 0 \qquad \text{for all } \theta \text{ and } t \\ \phi(r,\theta,0) &= (b-r)^2 \text{ for all } \theta \text{ and for } 0 \le r \le b \\ \frac{\partial\phi}{\partial t}(r,\theta,0) &= 0 \qquad \text{for all } \theta \text{ and for all } 0 \le r \le b \end{aligned}$$

which corresponds to the drum head being held fixed at its perimeter, and which was initially at rest with the prescribed initial displacement. You do not have to explicitly compute the integrals that provide the coefficients of the series solution.

- As an ansatz for the solution to this boundary value problem, we'll use a general linear combination of the Separation of Variables solution found in part (f). Thus, we set

$$(**) \qquad \phi(r,\theta,t) = \sum_{\lambda}\left(a_\lambda R_{0,\lambda}(r)\cos(c\lambda t) + b_\lambda R_{0,\lambda}(r)\sin(c\lambda t)\right)$$

Here's how one satisfies the first boundary condition. We have

$$R_{0,\lambda}(r) = J_0(\lambda r)$$

Now the Bessel functions $J_0(\lambda r)$ while not quite periodic, are at least quasi-periodic; meaning in particular that $J_0(x) = 0$ has an infinite number of solutions. Suppose we order these solutions as $0 < x_0 < x_1 < x_2 < \cdots$, then if we set

$$\lambda_i = \frac{x_i}{b}$$

then

$$R_{0,\lambda_i}(b) = J_0(\lambda_i b) = J_0\left(\frac{x_i}{b}b\right) = J_0(x_i) = 0$$

and so the first boundary condition

$$\phi(b,\theta,t) = 0 \qquad \forall\ \theta,\tau$$

will be automatically satisfied if restrict the sum over λ in (**) to be a sum over $\lambda_0, \lambda_1, \lambda_2, \ldots$. Note that chosing the λ_i in this way also guarantees our radial functions $R_{0,\lambda_i}(r)$ satisfy the orthogonality relations

$$\int_0^b R_{0,\lambda_i}(r) R_{0,\lambda_j}(r) r dr = 0 \qquad \text{if} \quad i \neq j$$

because the functions $R_{0,\lambda_i}(r)$ are solutions of the radial equation and satisfy the boundary conditions of the type appearing in part (d).

- Let us now set

$$\phi(r,\theta,t) = \sum_{i=0}^{\infty} R_{0,\lambda_i}(r) \left(a_\iota \cos(\lambda_i c t) + b_i \sin(\lambda_i c t)\right)$$

and impose the last two boundary conditions on $\phi(r,\theta,t)$.

$$\left(b - r^2\right) = \phi(r,\theta,0) = \sum_{i=0}^{\infty} a_i R_{0,\lambda_i}(r)$$

Multiplying this equation by $R_{0,\lambda_j}(r) r$ and integrating from 0 to b yields

$$\int_0^b \left(b - r^2\right) R_{0,\lambda_j}(r) r dr = \sum_{i=0}^{\infty} a_i \int_0^b R_{0,\lambda_i}(r) R_{0,\lambda_j}(r) r dr$$

But, in view of the orthogonality relations,

$$\int_0^b R_{0,\lambda_i}(r) R_{0,\lambda_j}(r) r dr = \begin{cases} \int_0^b R_{0,\lambda_j}(r)^2 r dr & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Thus the right hand side of (***) evaluates to

$$a_j \int_0^b R_{0,\lambda_j}(r)^2 r dr$$

and so

$$a_j = \frac{\int_0^b \left(b - r^2\right) R_{0,\lambda_j}(r) r dr}{\int_0^b R_{0,\lambda_j}(r)^2 r dr}$$

- The last boundary condition is

$$0 = \frac{\partial}{\partial t}\phi(r,\theta,0) = \sum_{i=0}^{\infty} (c\lambda_i) b_i R_{0,\lambda_i}(r)$$

This condition is readily satisfied by chosing $b_i = 0$ for $i = 0, 1, 2, \ldots$.

- We conclude that the solution to the PDE/BVP is given by

$$\phi(r,\theta,t) = \sum_{i=0}^{\infty} a_i R_{0,\lambda_i}(r) \cos(c\lambda_i t)$$

where
  - $\lambda_0, \lambda_1, \lambda_2, \ldots$ are the roots of $J_0(x) = 0$
  - $R_{0,\lambda_i}(r) = J_0(\lambda_i r)$
  - the coefficients a_i are given by

$$a_i = \frac{\int_0^b \left(b - r^2\right) R_{0,\lambda_i}(r) r dr}{\int_0^b R_{0,\lambda_i}(r)^2 r dr}$$