Math 4233

Homework Set 4

1. For each of the following PDEs, try using the method of separation of variables to replace the PDE by a pair of ODEs.

(a) $xu_{xx} + u_t = 0$

- Setting
$$u(x,t) = X(x)Y(t)$$
and plugging into the PDE we get
$$xX''(x)Y(t) + X(x)\dot{Y}(t) = 0$$
Dividing this equation by $X(x)Y(t)$ we get
$$x\frac{X''(x)}{X(x)} + \frac{\dot{Y}(t)}{Y(t)} = 0$$
or
$$x\frac{X''(x)}{X(x)} = -\frac{\dot{Y}(t)}{Y(t)}$$
Since the right hand side depends only on x and the left hand side depends only on t, we can conclude that both sides must be equal to a constant independent of x and t. We thus obtain
$$x\frac{X''(x)}{X(x)} = C = -\frac{\dot{Y}(t)}{Y(t)}$$
which leads to the following pair of ordinary differential equations
$$\begin{aligned} xX'' &= CX \\ \dot{Y} &= -CY \end{aligned}$$

(b) $u_{xx} + u_{xt} + u_t = 0$

- Setting $u(x,t) = X(x)Y(t)$, plugging into the PDE and then dividing by $X(x)Y(t)$ yields
$$\frac{X''(x)}{X(x)} + \frac{X'(x)\dot{Y}(t)}{XY} + \frac{\dot{Y}}{Y} = 0$$
This equation cannot be separated further.

(c) $tu_{xx} + xu_t = 0$

- Setting $u(x,t) = X(x)Y(t)$, plugging into the PDE and then dividing by $X(x)Y(t)$ yields
$$t\frac{X''(x)}{X(x)} + x\frac{\dot{Y}}{Y} = 0$$
If we now divide both sides by tx we get
$$\frac{X''(x)}{xX(x)} + \frac{\dot{Y}}{tY} = 0$$
or
$$\frac{X''(x)}{xX(x)} = -\frac{\dot{Y}}{tY}$$

Since the right hand side depends only on x and the left hand side depends only on t, we can conclude that both sides must be equal to a constant independent of x and t. We thus obtain

$$\frac{X''(x)}{xX(x)} = C = -\frac{\dot{Y}}{tY}$$

which leads to the following pair of ODEs:

$$X'' = CxX$$
$$\dot{Y} = -CtY$$

(d) $[p(x)\,u_x]_x - r(x)\,u_{tt} = 0$

- Setting $u(x,t) = X(x)\,Y(t)$, plugging into the PDE we get

$$\begin{aligned} 0 &= \frac{d}{dx}\left[p(x)\,X'(x)\,Y(t)\right] - r(x)\,X(x)\,\ddot{Y}(t) \\ &= Y(t)\,\frac{d}{dx}\left[p(x)\,X'(x)\right] - r(x)\,X(s)\,\ddot{Y}(t) \end{aligned}$$

Dividing both sides by and then dividing by $r(x)\,X(x)\,Y(t)$ yields

$$\frac{1}{r(x)\,X(x)}\frac{d}{dx}\left[p(x)\,X'(x)\right] - \frac{\ddot{Y}(t)}{Y(t)} = 0$$

or

$$\frac{1}{r(x)\,X(x)}\frac{d}{dx}\left[p(x)\,X'(x)\right] = \frac{\ddot{Y}(t)}{Y(t)}$$

Since the right hand side depends only on x and the left hand side depends only on t, we can conclude that both sides must be equal to a constant independent of x and t. We thus obtain

$$\frac{1}{r(x)\,X(x)}\frac{d}{dx}\left[p(x)\,X'(x)\right] = C = \frac{\ddot{Y}(t)}{Y(t)}$$

which leads to the following pair of ODEs:

$$\frac{d}{dx}\left[p(x)\,X'(x)\right] = Cr(x)\,X(x)$$
$$\ddot{Y}(t) = CY(t)$$

(Remark: the first differential equation is of Sturm-Liouville type - which we shall be studying shortly.)

(e) $u_{xx} + u_{yy} + xu = 0$

- Setting $u(x,t) = X(x)\,Y(t)$, plugging into the PDE we get

$$X''(x)\,Y(t) + X(x)\,\ddot{Y}(t) + xX(x)\,Y(t) = 0$$

Dividing this equation by $X(x)\,Y(t)$ yields

$$\frac{X''(x)}{X(x)} + \frac{\ddot{Y}(t)}{Y(t)} + x = 0$$

or

$$\frac{X''(x)}{X(x)} + x = -\frac{\ddot{Y}(t)}{Y(t)}$$

Since the right hand side depends only on x and the left hand side depends only on t, we can conclude that both sides must be equal to a constant independent of x and t. We thus obtain

$$\frac{X''(x)}{X(x)} + x = C = -\frac{\ddot{Y}(t)}{Y(t)}$$

which leads to the following pair of ODEs:

$$X'' + xX = CX$$
$$\ddot{Y} = -CY$$

2. Find the solution of the following heat conduction problem

(2a) $$4u_t - u_{xx} = 0 \quad , \quad 0 < x < 2 \quad , \quad t > 0$$

(2b) $$u(0,t) = 0$$

(2c) $$u(2,t) = 0$$

(2d) $$u(x,0) = 2\sin\left(\frac{\pi x}{2}\right) - \sin(\pi x) + 4\sin(2\pi x)$$

- We first use separation of variables to find a suitable family of solutions of the heat equation satisfying the first two boundary conditions. Thus, we look for functions of the form

(2e) $$u(x,t) = X(x)\,Y(t)$$

that will satisfy (2a), (2b) and (2c). Substituting (2e) into (2a) and dividing the result by $X(x)\,Y(t)$ we obtain

$$4\frac{\dot{Y}}{Y} = \frac{X''}{X}$$

Since the left hand side does not depend on x and the right hand side does not depend on t, we conclude that both sides must be equal to a constant, which we shall denote by $-\lambda^2$. We are thus led to

$$X'' = -\lambda^2 X \quad \Longrightarrow \quad X(x) = A\sin(\lambda x + \delta)$$
$$\dot{Y} = -\frac{\lambda^2}{4}Y \quad \Longrightarrow \quad Y(t) = Ce^{-\frac{\lambda^2}{4}t}$$

Imposing the boundary conditions at $x = 0$ on the expression (2e) we find

$$0 = u(0,t) = ACe^{-\frac{\lambda^2}{4}t}\sin(\delta) \quad \Longrightarrow \quad \delta = 0 \qquad \text{(for non-trivial solutions)}$$

Taking then $\delta = 0$ and imposing the boundary condition at $x = 2$ we find

$$0 = u(2,t) = ACe^{-\frac{\lambda^2}{4}t}\sin(2\lambda) \quad \Longrightarrow \quad \lambda = \frac{n\pi}{2} \quad , \quad n = 1, 2, \ldots \qquad \text{(for non-trivial solutions)}$$

Thus, any function of the form

$$\phi_n(x,t) = e^{-\left(\frac{n\pi}{2}\right)^2 t}\sin\left(\frac{n\pi}{2}x\right) \qquad , \qquad n = 1, 2, \ldots$$

will satisfy equations (2a) - (2c). Moreover, any linear combination of the functions ϕ_n will continue to satisfy equations (2a) - (2c). We thus set

(2f) $$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-\frac{1}{4}\left(\frac{n\pi}{2}\right)^2 t}\sin\left(\frac{n\pi}{2}x\right)$$

and try to choose the coefficients a_n so that the final boundary condition (2d) is satisfied. Plugging (2f) into (2d) we obtain

$$2\sin\left(\frac{\pi x}{2}\right) - \sin(\pi x) + 4\sin(2\pi x) = \sum_{n=1}^{\infty} a_n e^{-\frac{1}{4}\left(\frac{n\pi}{2}\right)^2 0}\sin\left(\frac{n\pi}{2}x\right) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi}{2}x\right)$$

From this we conclude that the coefficients a_n should coincide with coefficients of $\sin\left(\frac{n\pi}{2}x\right)$ in the Fourier-sine expansion of the function on the right hand side on the interval $[0,2]$. Thus, we set

$$\begin{aligned} a_n &= \frac{2}{L}\int_0^L \left(2\sin\left(\frac{\pi x}{2}\right) - \sin(\pi x) + 4\sin(2\pi x)\right)\sin\left(\frac{n\pi}{L}x\right)dx \\ &= \int_0^2 \left(2\sin\left(\frac{\pi x}{2}\right) - \sin(\pi x) + 4\sin(2\pi x)\right)\sin\left(\frac{n\pi}{2}x\right)dx \end{aligned}$$

By the orthogonality properties of the Fourier-sine functions

$$\frac{2}{L}\int_0^L \sin\left(\frac{m\pi}{L}x\right)\sin\left(\frac{n\pi}{L}x\right)dx = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$$

we find

$$a_n = \begin{cases} 2 & \text{if } n = 1 \\ -1 & \text{if } n = 2 \\ 4 & \text{if } n = 4 \\ 0 & \text{otherwise} \end{cases}$$

Thus,

$$u(x,t) = 2e^{-\frac{\pi^2}{8}t}\sin\left(\frac{\pi}{2}x\right) - e^{-\frac{\pi^2}{4}^2 t}\sin(\pi x) + 4e^{-\pi^2 t}\sin(2\pi x)$$

3. Find the solution of

$$\begin{aligned} 4u_t - u_{xx} &= 0 \qquad , \qquad 0 < x < 2 \quad , \quad t > 0 \\ u(0,t) &= 2 \\ u(2,t) &= -2 \\ u(x,0) &= 2\cos(\pi x) \end{aligned}$$

- Because of the non-homogeneous boundary conditions at $x = 0$ and $x = 2$, we first construct a time-independent (steady-state) solution that will satisfy these boundary conditions. Suppose

$$u(x,t) = u_{ss}(x)$$

Plugging this into the heat equation we find

$$\begin{aligned} 4\frac{\partial}{\partial t}u_{ss}(x) - \frac{\partial^2}{\partial x^2}u_{ss}(x) = 0 \quad &\Longrightarrow \quad \frac{d^2 u_{ss}}{dx^2} = 0 \\ &\Longrightarrow \quad u_{ss} = Ax + B \end{aligned}$$

Imposing the boundary conditions at $x = 0$ and $x = 2$

$$\left.\begin{aligned} u_{ss}(0) &= 2 \\ u_{ss}(2) &= -2 \end{aligned}\right\} \quad \Longrightarrow \quad u_{ss} = -2x + 2$$

Now we set

(*)
$$u(x,t) = u_{ss}(x) + \tau(x,t)$$

where $\tau(x,t)$ is an auxiliary function corresponding to the (time-dependent) discrepancy between the actual solution of the given boundary value problem and the steady-state solution u_{ss}. Imposing the boundary conditions on $u(x,t)$ we find

$$\begin{aligned} 2 = u(0,t) &= u_{ss}(0) + \tau(0,t) = 2 + \tau(0,t) \\ \Longrightarrow &\quad \tau(0,t) = 0 \\ -2 = u(2,t) &= u_{ss}(2) + \tau(2,t) = -2 - \tau(2,t) \\ \Longrightarrow &\quad \tau(2,t) = 0 \\ 2\cos(\pi x) = u(x,0) &= u_{ss}(x) + \tau(x,0) = -2x + 2 + \tau(x,0) \\ \Longrightarrow &\quad \tau(x,0) = -2x + 2 + 2\cos\pi x \end{aligned}$$

Since $\frac{\partial u_{ss}}{\partial t} = 0 = \frac{\partial^2 u_{ss}}{\partial x^2}$, when we plug (*) into the heat equation for $u(x,t)$, we find

$$4\frac{\partial\tau}{\partial t} - \frac{\partial^2\tau}{\partial x^2} = 0$$

In other words, $\tau(x,t)$ must satisfy the following boundary value problem:

$$\begin{aligned}4\frac{\partial \tau}{\partial t}-\frac{\partial^2\tau}{\partial x^2}&=0\\ \tau(0,t)&=0\\ \tau(2,t)&=0\\ \tau(x,0)&=-2x+2+2\cos\pi x\end{aligned}$$

In the previous problem we worked out the solution to a similar system, the only difference being the function that appears on the right hand side of the last boundary conditions. As before, separation of variables and boundary conditions at $x=0$ and $x=2$ will lead us to the following ansatz for the solution of the BVP for $\tau(x,t)$

$$\tau(x,t)=\sum_{n=1}^{\infty}a_n\sin\left(\frac{n\pi}{2}x\right)$$

If we now impose the boundary condition at $t=0$ we see

$$-2x+2+2\cos(\pi x)=\sum_{n=1}^{\infty}a_n\sin\left(\frac{n\pi}{2}x\right)dx$$

To determine the coefficients a_n we multiply both sides by $\sin\left(\frac{m\pi}{2}x\right)$ and integrate from $x=0$ to $x=2$

$$\int_0^2(2-2x+2\cos(\pi x))\sin\left(\frac{m\pi}{2}x\right)dx=\sum_{n=1}^{\infty}\int_0^2 a_n\sin\left(\frac{n\pi}{2}x\right)\sin\left(\frac{m\pi}{2}x\right)dx=\sum_{n=1}^{\infty}a_n\delta_{m,n}=a_m$$

We just need to compute

$$a_m=\int_0^2(2-2x+2\cos(\pi x))\sin\left(\frac{m\pi}{2}x\right)dx$$

Now

$$\int_0^2 2\sin\left(\frac{m\pi}{2}x\right)dx=2\left(-\frac{2}{m\pi}\cos\left(\frac{m\pi}{2}\right)x\right)\Big|_0^2=\frac{4}{m\pi}(1-\cos(m\pi))=\begin{cases}0 & \text{if } m \text{ is even}\\ \frac{8}{m\pi} & \text{if } m \text{ is odd}\end{cases}$$

$$\begin{aligned}-2\int_0^2 x\sin\left(\frac{m\pi}{2}x\right)dx&=-\frac{2}{\left(\frac{m\pi}{2}\right)^2}\left(\sin\left(\frac{m\pi}{2}x-\left(\frac{m\pi}{2}x\right)\cos\left(\frac{m\pi}{2}\right)x\right)\right)\Big|_0^2\\ &=\frac{8}{m^2\pi^2}\cos(m\pi)\\ &=-(1)^m\frac{8}{m^2\pi^2}\end{aligned}$$

$$\begin{aligned}2\int_0^2\cos(\pi x)\sin\left(\frac{m\pi}{2}x\right)dx&=-\frac{1}{2}\frac{1}{\pi+\frac{m\pi}{2}}\cos\left(\left(\pi+\frac{m\pi}{2}\right)x\right)-\frac{1}{2}\frac{1}{-\pi+\frac{m\pi}{2}}\cos\left(\left(-\pi+\frac{m\pi}{2}\right)x\right)\Big|_0^2\\ &=-\frac{2m}{\pi(m^2-4)}(\cos(m\pi-1))=\begin{cases}0 & \text{if } m \text{ is even}\\ \frac{4}{\pi(m^2-4)} & \text{if } m \text{ is odd}\end{cases}\end{aligned}$$

Thus,

$$a_m=\begin{cases}\frac{8}{m^2\pi^2} & \text{if } m \text{ is even}\\ -\frac{8}{m^2\pi^2}+\frac{8}{m\pi}+\frac{4}{\pi(m^2-4)} & \text{if } m \text{ is odd}\end{cases}$$

4. Show that the wave equation

$$(*)\qquad u_{tt}-a^2u_{xx}=0$$

can be reduced to the form

$$u_{\xi\eta}=0$$

by a change for variables $\xi = x - at$, $\eta = x + at$. Conclude that the any solution of (*) can be written as

$$u(x,t) = \phi(x - at) + \psi(x + at) \quad .$$

- We have

$$\left.\begin{array}{c} \xi = x - at \\ \eta = x + at \end{array}\right\} \longleftrightarrow \left\{ \begin{array}{c} x = \frac{1}{2}\xi - \frac{1}{2}\eta \\ t = \frac{1}{2a}\eta - \frac{1}{2a}\xi \end{array}\right.$$

  and we have

$$\begin{aligned} \frac{\partial}{\partial x} &= \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} \\ \frac{\partial}{\partial t} &= \frac{\partial \xi}{\partial t}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial t}\frac{\partial}{\partial \eta} = -a\frac{\partial}{\partial \xi} + a\frac{\partial}{\partial \eta} \end{aligned}$$

  and so

$$\begin{aligned} \frac{\partial^2}{\partial t^2} &= \left(-a\frac{\partial}{\partial \xi} + a\frac{\partial}{\partial \eta}\right)\left(-a\frac{\partial}{\partial \xi} + a\frac{\partial}{\partial \eta}\right) = a^2\frac{\partial^2}{\partial \xi^2} - 2a^2\frac{\partial^2}{\partial \eta \partial \xi} + a^2\frac{\partial^2}{\partial \eta^2} \\ \frac{\partial^2}{\partial x^2} &= \left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right)\left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right) = \frac{\partial^2}{\partial \xi^2} + 2\frac{\partial^2}{\partial \eta \partial \xi} + \frac{\partial^2}{\partial \eta^2} \end{aligned}$$

  Thus,

$$\begin{aligned} \left(\frac{\partial^2}{\partial t^2} - a^2\frac{\partial^2}{\partial x^2}\right)u &= \left(a^2\frac{\partial^2}{\partial \xi^2} - 2a^2\frac{\partial^2}{\partial \eta \partial \xi} + a^2\frac{\partial^2}{\partial \eta^2} - a^2\left(\frac{\partial^2}{\partial \xi^2} + 2\frac{\partial^2}{\partial \eta \partial \xi} + \frac{\partial^2}{\partial \eta^2}\right)\right)u \\ &= \left(-4a^2\frac{\partial^2}{\partial \eta \partial \xi}\right)u \end{aligned}$$

  and so

$$u_{tt} - a^2 u_{xx} = 0 \quad \Longrightarrow \quad -4a^2 u_{\xi\eta} = 0 \quad \Longrightarrow \quad u_{\xi\eta} = 0$$

- *A priori* a solution of $u_{\xi\eta} = 0$ could consist of terms $\phi(\xi)$ that depend only on ξ, terms $\psi(\eta)$ that depend only on η and terms $\sigma(\xi,\eta)$ that depend on both ξ and η;

$$u(\xi,\eta) = \phi(\xi) + \psi(\eta) + \sigma(\xi,\eta)$$

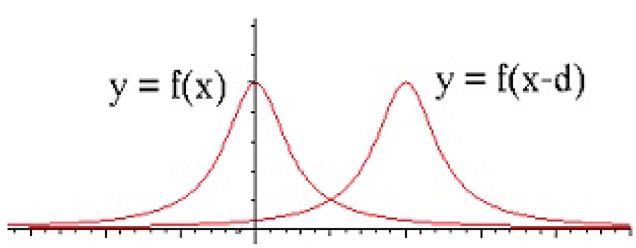
  The condition $u_{\xi\eta} = 0$ then implies that

$$0 = \frac{\partial}{\partial \xi}\frac{\partial}{\partial \eta}\left(\phi(\xi) + \psi(\eta) + \sigma(\xi,\eta)\right) = 0 + 0 + \frac{\partial}{\partial \xi}\frac{\partial}{\partial \eta}\sigma(\xi,\eta)$$

  by assumption each term of $\sigma(\xi,\eta)$ depends non-trivially on both ξ and η, and so unless $\sigma(\xi,\eta) = 0$, we'll have $u_{\xi\eta} \neq 0$. We conclude that any solution of the wave equation must be of the form

(**) $$u(\xi,\eta) = \phi(\xi) + \psi(\eta) = \phi(x - at) + \psi(x + at)$$

- Note also that the graph of a function $f(x - d)$ as a function of x looks exactly the graph of $f(x)$ translated a distance d down the x-axis.



  Because of this for fixed t the graph of $\phi(x - at)$ will look like the graph of $\phi(x)$ translated a distance at down the x-axis. Letting t now vary, one can interprete the graph of the solution $\phi(x - at)$ as corresponding to a certain wave packet of shape $y = \phi(x)$ propagating down the x-axis with velocity a (since $d = at$ will be the graph's displacement at time t). Similarly, the graph of the solution $\psi(x + at)$ can be interpreted as a certain wave packet of shape $y = \psi(x)$ propagating down the

x-axis in the opposite direction. Thus, via (**) every solution of the wave equation can be thought of as the superposition of two wave packets one moving to the right and one moving to the left.

5. Find the solution of Laplace's equation

$$u_{xx} + u_{yy} = 0$$

satisfying the boundary conditions

$$\begin{array}{lcl} u(x,0) = 0 & , & u(x,b) = g(x) \\ u(0,y) = 0 & , & u(a,y) = 0 \end{array}$$

- Setting $u(x,y) = X(x)Y(y)$ and separating variables we quickly obtain

$$\begin{array}{lcl} X''(x) = -\lambda^2 X(x) & \Longrightarrow & X(x) = A\sin(\lambda x + \delta) \\ Y''(y) = \lambda^2 Y(y) & \Longrightarrow & Y(y) = c_1 \sinh(\lambda y) + c_2 \cosh(\lambda y) \end{array}$$

(We choose the separation constant to be a negative number $-\lambda^2$ with a little bit of foresight. If we had chosen it to be positive, the functions $X(x)$ would not be sinusoidal functions of x and we would not be able to easily satisfy the boundary conditions at $x = 0$ and $x = a$.) The boundary conditions at $x = 0$ require

$$0 = u(0,y) = X(0)Y(y) \quad \Longrightarrow \quad X(0) = 0 \quad \Longrightarrow \quad A\sin(\delta) = 0 \quad \Longrightarrow \quad \delta = 0$$

and then taking $\delta = 0$ and imposing the boundary condition at $x = a$ we find

$$0 = u(a,y) = X(a)Y(y) \quad \Longrightarrow \quad X(a) = 0 \quad \Longrightarrow \quad A\sin(\lambda a + 0) = 0 \quad \Longrightarrow \quad \lambda = \frac{n\pi}{a} \ , \ n = 1,2,3,\ldots$$

The boundary condition at $y = 0$ require

$$0 = u(x,0) = X(x)Y(0) \quad \Longrightarrow \quad Y(0) = 0 \quad \Longrightarrow \quad c_2 = 0$$

and so any function of the form

$$\sin\left(\frac{n\pi}{a}x\right)\sinh\left(\frac{n\pi}{a}y\right)$$

will satisfy Laplace's equation and three of the boundary conditions. So also will any linear combination of these functions. So we set

$$u(x,y) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{a}x\right)\sinh\left(\frac{n\pi}{a}y\right)$$

and impose the last boundary condition

$$(*) \qquad g(x) = u(x,b) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{a}x\right)\sinh\left(\frac{n\pi}{a}b\right)$$

On the other hand, so long as $g(x)$ is continuous and differntiable, it will have a Fourier-sine expansion of the form

$$(**) \qquad g(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{a}x\right) \quad , \quad b_n := \frac{2}{a}\int_0^a g(x)\sin\left(\frac{n\pi}{a}x\right)dx$$

Comparing the coefficients of (linearly independent functions) $\sin\left(\frac{n\pi}{a}x\right)$ on the right hand sides of (*) and (**) we conclude

$$b_n = c_n \sinh\left(\frac{n\pi}{a}b\right)$$

or

$$c_n = \frac{b_n}{\sinh\left(\frac{n\pi}{a}b\right)} = \frac{2}{a\sinh\left(\frac{n\pi}{a}b\right)}\int_0^a g(x)\sin\left(\frac{n\pi}{a}x\right)dx$$

6. Express the 2-dimensional Laplace equation

$$u_{xx} + u_{yy} = 0$$

in terms of polar coordinates (r, θ) and use separation of variables to reduce it to the solution of a pair of ordinary differential equations.

- We have

$$\left.\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array}\right\} \quad \Longleftrightarrow \quad \left\{\begin{array}{l} r = \sqrt{x^2+y^2} \\ \theta = \tan^{-1}(y/x) \end{array}\right.$$

According to the multi-variable chain rule

$$\begin{aligned} \frac{\partial}{\partial x} &= \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial y} &= \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta} \end{aligned}$$

Now

$$\begin{aligned} \frac{\partial r}{\partial x} &= \frac{x}{\sqrt{x^2+y^2}} = \frac{r\cos(\theta)}{r} = \cos(\theta) \\ \frac{\partial r}{\partial y} &= \frac{y}{\sqrt{x^2+y^2}} = \frac{r\sin(\theta)}{r} = \sin(\theta) \\ \frac{\partial \theta}{\partial x} &= -\frac{y}{x^2+y^2} = -\frac{r\sin(\theta)}{r^2} = -\frac{1}{r}\sin(\theta) \\ \frac{\partial \theta}{\partial y} &= \frac{x}{x^2+y^2} = \frac{r\cos(\theta)}{r^2} = \frac{1}{r}\cos(\theta) \end{aligned}$$

and so

$$\begin{aligned} \frac{\partial}{\partial x} &= \cos(\theta)\frac{\partial}{\partial r} - \frac{1}{r}\sin(\theta)\frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial y} &= \sin(\theta)\frac{\partial}{\partial r} + \frac{1}{r}\cos(\theta)\frac{\partial}{\partial \theta} \end{aligned}$$

We then have

$$\begin{aligned} \left(\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial x}\right) &= \left(\cos(\theta)\frac{\partial}{\partial r} - \frac{1}{r}\sin(\theta)\frac{\partial}{\partial \theta}\right)\left(\cos(\theta)\frac{\partial}{\partial r} - \frac{1}{r}\sin(\theta)\frac{\partial}{\partial \theta}\right) \\ &= \left(\cos(\theta)\frac{\partial}{\partial r}\right)\left(\cos(\theta)\frac{\partial}{\partial r}\right) - \left(\cos(\theta)\frac{\partial}{\partial r}\right)\left(\frac{1}{r}\sin(\theta)\frac{\partial}{\partial \theta}\right) \\ &- \left(\frac{1}{r}\sin(\theta)\frac{\partial}{\partial \theta}\right)\left(\cos(\theta)\frac{\partial}{\partial r}\right) + \left(\frac{1}{r}\sin(\theta)\frac{\partial}{\partial \theta}\right)\left(\frac{1}{r}\sin(\theta)\frac{\partial}{\partial \theta}\right) \\ &= \cos^2(\theta)\frac{\partial^2}{\partial r^2} - \cos(\theta)\sin(\theta)\frac{\partial}{\partial r}\left(\frac{1}{r}\right)\frac{\partial}{\partial \theta} \\ &- \frac{1}{r}\sin(\theta)\left(\frac{\partial}{\partial \theta}\cos\theta\right)\frac{\partial}{\partial r} + \frac{1}{r^2}\sin(\theta)\left(\frac{\partial}{\partial \theta}\sin(\theta)\right)\frac{\partial}{\partial \theta} \\ &= \cos^2(\theta)\frac{\partial^2}{\partial r^2} - \cos(\theta)\sin(\theta)\left(-\frac{1}{r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\frac{\partial}{\partial \theta} \\ &- \frac{1}{r}\sin(\theta)\left(-\sin(\theta) + \cos(\theta)\frac{\partial}{\partial \theta}\right)\frac{\partial}{\partial r} \\ &+ \frac{1}{r^2}\sin(\theta)\left(\cos(\theta) + \sin(\theta)\frac{\partial}{\partial \theta}\right)\frac{\partial}{\partial \theta} \end{aligned}$$

and

$$\begin{aligned}\left(\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial y}\right) &= \left(\sin(\theta)\frac{\partial}{\partial r}+\frac{1}{r}\cos(\theta)\frac{\partial}{\partial\theta}\right)\left(\sin(\theta)\frac{\partial}{\partial r}+\frac{1}{r}\cos(\theta)\frac{\partial}{\partial\theta}\right)\\
&= \sin^2\theta\frac{\partial^2}{\partial r^2}+\sin\theta\cos\theta\frac{\partial}{\partial r}\left(\frac{1}{r}\right)\frac{\partial}{\partial\theta}\\
&+\frac{1}{r}\cos\theta\left(\frac{\partial}{\partial\theta}\sin\theta\right)\frac{\partial}{\partial r}+\frac{1}{r^2}\cos(\theta)\left(\frac{\partial}{\partial\theta}\cos\theta\right)\frac{\partial}{\partial\theta}\\
&= \sin^2\theta\frac{\partial^2}{\partial r^2}+\sin\theta\cos\theta\left(-\frac{1}{r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right)\frac{\partial}{\partial\theta}\\
&+\frac{1}{r}\cos\theta\left(\cos(\theta)+\sin(\theta)\frac{\partial}{\partial\theta}\right)\frac{\partial}{\partial r}\\
&+\frac{1}{r^2}\cos(\theta)\left(-\sin(\theta)+\cos(\theta)\frac{\partial}{\partial\theta}\right)\frac{\partial}{\partial\theta}\end{aligned}$$

Thus,

$$\begin{aligned}\left(\frac{\partial}{\partial x}\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\frac{\partial}{\partial y}\right) &= \left(\sin^2\theta+\cos^2\theta\right)\frac{\partial^2}{\partial r^2}+0\cdot\left(-\frac{1}{r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right)\frac{\partial}{\partial\theta}\\
&-\frac{1}{r}\sin(\theta)\left(-\sin(\theta)+\cos(\theta)\frac{\partial}{\partial\theta}\right)\frac{\partial}{\partial r}+\frac{1}{r}\cos\theta\left(\cos(\theta)+\sin(\theta)\frac{\partial}{\partial\theta}\right)\frac{\partial}{\partial r}\\
&+\frac{1}{r^2}\sin(\theta)\left(\cos(\theta)+\sin(\theta)\frac{\partial}{\partial\theta}\right)\frac{\partial}{\partial\theta}+\frac{1}{r^2}\cos(\theta)\left(-\sin(\theta)+\cos(\theta)\frac{\partial}{\partial\theta}\right)\frac{\partial}{\partial\theta}\\
&= \frac{\partial^2}{\partial r^2}+\frac{1}{r}\left(\sin^2\theta+\cos^2\theta\right)\frac{\partial}{\partial\theta}\frac{\partial}{\partial r}+\frac{1}{r^2}\left(\sin^2\theta+\cos^2\theta\right)\frac{\partial^2}{\partial\theta^2}\\
&= \frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\end{aligned}$$

Laplace's equation in polar coordinates is thus

(***)
$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)u=0$$

- We'll now try separation of variables for Laplace's equation in polar coordinates. Setting

$$u(r,\theta)=R(r)T(\theta)$$

and plugging into (***) we get

$$R''T+\frac{1}{r}TR'+\frac{1}{r^2}RT''=0$$

Dividing through by RT we get

$$\frac{R''}{R}+\frac{R'}{rR}+\frac{1}{r^2}\frac{T''}{T}=0$$

or which after multiplying by r^2 leads to

$$r^2\frac{R''}{R}+r\frac{R'}{R}=-\frac{T''}{T}$$

Setting the separation constant equal to a constant λ^2 we arrive at

$$\begin{aligned}r^2R''+rR'&=\lambda^2R\\ T''&=-\lambda^2T\end{aligned}$$