Math 4233
Solutions to Homework Set 3

1. The equation of motion for a spring-mass system with damping is

$$m\frac{dx^2}{dt^2} + c\frac{dx}{dt} + kx = 0$$

where m, c, k are positive constants (mass, damping coefficient and spring force constant).

(a) Write this equation as a system of two first order equations for $u_1 = x(t)$ and $u_2(t) = \frac{du}{dt}$.

- From the definition of u_2 we have

$$\frac{du_1}{dt} = u_2$$

and substituting $\frac{du_2}{dt}$ for $\frac{dx^2}{dt^2}$, u_2 for $\frac{dx}{dt}$ and u_1 for x in the $\frac{1}{m}$ times original differential equation, we obtain

$$\frac{du_2}{dt} = -\frac{k}{m}u_1 - \frac{c}{m}u_2$$

These differential equations for u_1 and u_2 can now be consolidated as a 2×2 linear system

$$\frac{d}{dt}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

(b) Show that $u_1 = 0$, $u_2 = 0$ is a critical point and analyze the nature and stability of the critical point as a function of the parameters m, c and k.

- The nature of the critical points of a linear system $\frac{d}{dt}\mathbf{u} = \mathbf{A}\mathbf{u}$ at $(0,0)$ is determined by the nature of the eigenvalues of the matrix $\mathbf{A}$. In the case at hand, the characteristic equation for $\mathbf{A} = \left[\begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix}\right]$ is

$$0 = \det(\mathbf{A} - \lambda) = -\lambda\left(-\frac{c}{m} - \lambda\right) + \frac{k}{m} = \lambda^2 + \frac{c}{m}\lambda + \frac{k}{m}$$

$$\Rightarrow \quad \lambda_\pm = \frac{-\frac{c}{m} \pm \sqrt{\left(\frac{c}{m}\right)^2 - 4\frac{k}{m}}}{2} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}$$

  – We first note that

$$\frac{k}{m} > \left(\frac{c}{2m}\right)^2 \quad \Rightarrow \quad \lambda_\pm \text{ is complex with negative real part}$$

  and so when this inequality holds, since both eigenvalues have negative real part, the critical point at $(0,0)$ will be *stable and asymptotically stable.*

  – When

$$\frac{k}{m} = \left(\frac{c}{2m}\right)^2$$

  we have only one eigenvalue $-\frac{c}{2m}$ and it is negative (our assumption on c and m is that they are positive). Thus, the critical point at $(0,0)$ will be *stable and asymptotically stable*

  – Finally, we have the possibility that

$$\left(\frac{c}{2m}\right)^2 > \frac{k}{m}$$

  In this case, the eigenvalues will also be negative; however, it is not quite so obvious. What is apparent is that $\lambda_\pm$ are at least real. The question though is: can

$$\lambda_+ = -\frac{c}{2m} + \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}$$

be positive? However, because $\sqrt{x}$ is an monotonically increasing function of x,t $\sqrt{x-h}$ is always less than $\sqrt{x}$ (so long as $x-h>0$). But then

$$\sqrt{\left(\frac{c}{2m}\right)^2-\frac{k}{m}}<\sqrt{\left(\frac{c}{2m}\right)^2}=\frac{c}{2m} \quad \Rightarrow \quad \lambda_+<0$$

And so in this case, both eigenvalues are negative real numbers and $(0,0)$ is *stable and asympotically stable.*

2. Consider the linear system

$$\frac{dx}{dt}=a_{11}x+a_{12}y$$
$$\frac{dy}{dt}=a_{21}x+a_{22}y$$

where a_{11}, a_{12}, a_{21}, and a_{22} are real constants and the corresponding matrix is diagonalizable.

$$p=a_{11}+a_{22}=trace\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{pmatrix}$$
$$q=a_{11}a_{22}-a_{12}a_{21}=\det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{pmatrix}$$

(a) Show that the critical point $(0,0)$ is a node if $q>0$ and $p^2-4q>0$

(b) Show that the critical point $(0,0)$ is a saddle point if $q<0$

(c) Show that the critical point $(0,0)$ is a spiral point if $p\neq 0$ and $p^2-4q<0$

(d) Show that the critical point $(0,0)$ is a stable center point if $p=0$ and $q>0$.

Hint: Note first that in terms of the eigenvalues r_1, r_2 of the coefficient matrix, $p=r_1+r_2$ and $r_1r_2=q$.

- Since $\mathbf{A}$ is diagonalizable, it has two linearly independent eigenvectors, say $\mathbf{v}_1$ and $\mathbf{v}_2$. Let r_1 and r_2 be the corresponding eigenvalues. Then we have

$$\mathbf{D}=\begin{pmatrix} r_1 & 0 \\ 0 & r_2\end{pmatrix}=\mathbf{C}^{-1}\mathbf{AC}$$

where $\mathbf{C}$ is the 2×2 formed by using $\mathbf{v}_1$ and $\mathbf{v}_2$ as columns. We have

$$r_1+r_2=trace\,(\mathbf{D})=trace\left(\mathbf{C}^{-1}\mathbf{AC}\right)=trace\left(\mathbf{ACC}^{01}\right)=trace\,(\mathbf{A})\equiv p$$

and

$$r_1r_2=\det(\mathbf{D})=\det\left(\mathbf{C}^{-1}\mathbf{AC}\right)=\det\left(\mathbf{C}^{-1}\right)\det(\mathbf{A})\det(\mathbf{C})$$
$$=\frac{1}{\det(\mathbf{C})}\det(\mathbf{A})\det(\mathbf{C})=\det(\mathbf{A})\equiv q$$

Solving the relations

$$r_1+r_2=p$$
$$r_1r_2=q$$

for r_1 and r_2 yields

$$r_1=\frac{p}{2}\pm\frac{\sqrt{p^2-4q}}{2}$$
$$r_2=p-r_1=\frac{p}{2}\mp\frac{\sqrt{p^2-4q}}{2}$$

Note that if we choose the "+ root" for r_1, then r_2 is the "$-$" root:

$$r_1 = \frac{p}{2} + \frac{\sqrt{p^2 - 4q}}{2} \quad \Rightarrow \quad r_2 = \frac{p}{2} - \frac{\sqrt{p^2 - 4q}}{2}$$

Thus, the ambiguity in sign for r_1 amounts to choice of which eigenvalue we call r_1. Consulting Table 9.1.1 in the text, we conclude:

- (a) In order to have a node, both eigenvalues of $\mathbf{A}$ have to be real numbers with the same sign. This amounts to demanding $p^2 - 4q \geq 0$ (so that r_1 and r_2 are real), and requiring $q = r_1 r_2 > 0$.
- (b) To have a saddle point, we need both eigenvalues to be real, but to have opposite signs. This leads to the requirements $p^2 - 4q > 0$ and $q = r_1 r_2 < 0$.
- (c) A spiral center occurs if the eigenvalues are complex and have non-zero real part. This leads to the requirements $p^2 - 4q < 0$ and $p < 0$.
- (d) A stable center occures when the eigenvalues are pure imaginary. This leads to the requirement $p = 0$ and $p^2 - 4q < 0$; or $p = 0$ and $q > 0$.

4. For each nonlinear system below, verify that $(0,0)$ is a critical point and that the system is locally linear about $(0,0)$. Discuss the stability of the critical point $(0,0)$ by examining the corresponding linear system.

(a)

$$\begin{aligned} \frac{dx}{dt} &= x - y^2 \\ \frac{dy}{dt} &= x - 2y + x^2 \end{aligned}$$

- Obviously, the right hand sides of these ODEs vanish when $x = 0$ and $y = 0$; and so $(0,0)$ is a critical point for the system. The coefficient matrix for the corresponding linearization is

$$\mathbf{A} = \left(\begin{array}{cc} \frac{\partial}{\partial x}\left(x - y^2\right) & \frac{\partial}{\partial y}\left(x - y^2\right) \\ \frac{\partial}{\partial x}\left(x - 2y + x^2\right) & \frac{\partial}{\partial y}\left(x - 2y + x^2\right) \end{array} \right) \Bigg|_{\substack{x=0 \\ y=0}} = \left(\begin{array}{cc} 1 & -2y \\ 1 + 2x & -2 \end{array} \right) \Bigg|_{\substack{x=0 \\ y=0}} = \left(\begin{array}{cc} 1 & 0 \\ 1 & -2 \end{array} \right)$$

We have

$$\det\left(\mathbf{A} - \lambda \mathbf{I}\right) = (1 - \lambda)(-2 - \lambda)$$

and so we have two real eigenvalues $\lambda = 1$ and $\lambda = -2$. $\mathbf{A}$ having two real eigenvalues of opposite signs means the system has an *unstable saddle point* at $(0,0)$.

(b)

$$\begin{aligned} \frac{dx}{dt} &= x + y^2 \\ \frac{dy}{dt} &= x + y \end{aligned}$$

- Again the system has an obvious critical point at $(0,0)$. The coefficient matrix for the corresponding linearization is

$$\mathbf{A} = \left(\begin{array}{cc} \frac{\partial}{\partial x}\left(x + y^2\right) & \frac{\partial}{\partial y}\left(x + y^2\right) \\ \frac{\partial}{\partial x}\left(x + y\right) & \frac{\partial}{\partial y}\left(x + y\right) \end{array} \right) \Bigg|_{\substack{x=0 \\ y=0}} = \left(\begin{array}{cc} 1 & 2y \\ 1 & 1 \end{array} \right) \Bigg|_{\substack{x=0 \\ y=0}} = \left(\begin{array}{cc} 1 & 0 \\ 1 & 1 \end{array} \right)$$

We have

$$\det\left(\mathbf{A} - \lambda \mathbf{I}\right) = (1 - \lambda)(1 - \lambda)$$

Since $\mathbf{A}$ has a single eigenvalue which is positive we conclude $(0,0)$ is a *proper or improper node* and is *not stable.*

5. For each of the following systems carry out the following steps.

(i) Identify the critical points.
(ii) For each critical point $\mathbf{c}$, identify the corresponding linear system. Write down the general solution of these linear systems and discuss the stability of the solutions near the critical solution $\mathbf{x}(t) = \mathbf{c}$.
(iii) Plot the direction field of the original system and discuss the evolution of the system for various initial conditions.

- Here we have

$$\mathbf{F}(x,y) = \begin{bmatrix} x(1-x-y) \\ y(1.5-y-x) \end{bmatrix} \quad \Longrightarrow$$

$$\mathbf{0} = \mathbf{F}(x,y) \quad \Longrightarrow \begin{cases} x=0\ ,\ y=0 \\ x=0\ ,\ y=1.5 \\ x=1\ ,\ y=0 \end{cases}$$

Thus, there are only three critical points. We have

$$\frac{\mathbf{dF}}{\mathbf{dx}} := \begin{bmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{bmatrix} = \begin{bmatrix} 1-2x-y & -x \\ -y & 1.5-x-2y \end{bmatrix}$$

  - Linear system at $(0,0)$:

$$\mathbf{y}' = \mathbf{A}\mathbf{y}$$

where

$$\mathbf{A} = \left.\frac{\mathbf{dF}}{\mathbf{dx}}\right|_{(0,0)} = \begin{bmatrix} 1 & 0 \\ 0 & 1.5 \end{bmatrix}$$

The matrix $\mathbf{A}$, being diagonal, obviously has eigenvalues 1 and 1.5 and has $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ corresponding eigenvectors. Thus, the general solution of the linearization about $(0,0)$ will look like

$$\mathbf{y}(t) = c_1 e^t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{1.5t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Since both eigenvalues are positive, the origin will be an unstable critical point.

  - Linear system at $(0,1.5)$:

$$\mathbf{A} = \begin{bmatrix} -0.5 & 0 \\ -1.5 & -1.5 \end{bmatrix}$$

$$\text{eigenvalues/eigenvectors} \quad \to \quad \begin{cases} r_1 = -0.5 \quad , \quad \xi_1 = \begin{bmatrix} 1 \\ -1.5 \end{bmatrix} \\ r_2 = -0.5 \quad , \quad \xi_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{cases}$$

Since both eigenvalues are negative, the critical point at $(0,1.5)$ is asymptotically stable.

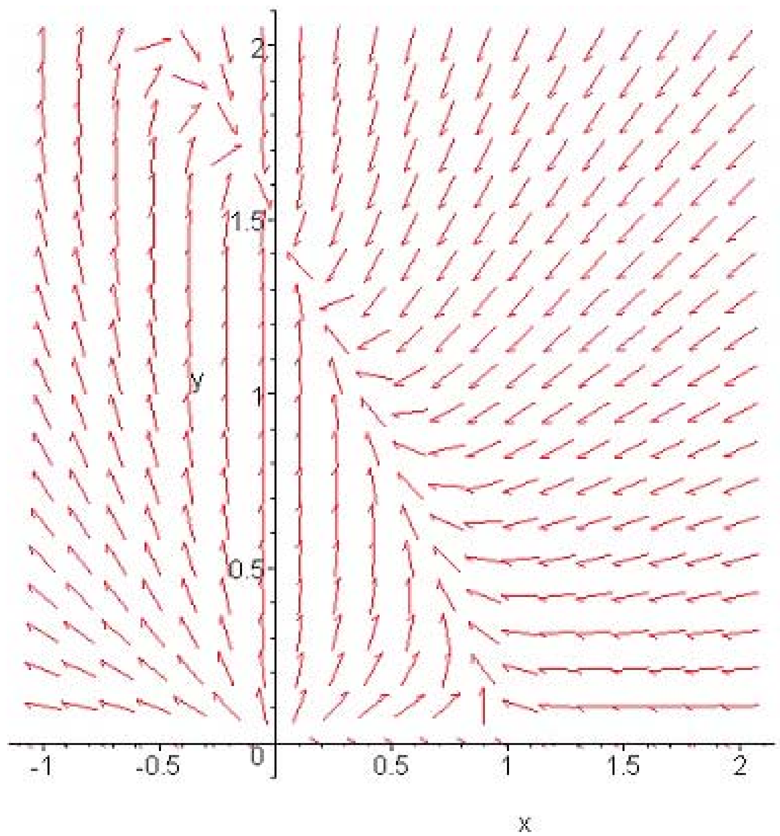
  - Linear system at $(1,0)$:

$$\mathbf{A} = \begin{bmatrix} -1 & -1 \\ 0 & 0.5 \end{bmatrix}$$

$$\text{eigenvalues/eigenvectors} \quad \to \quad \begin{cases} r_1 = -1 \quad , \quad \xi_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ r_2 = -0.5 \quad , \quad \xi_2 = \begin{bmatrix} -2 \\ 3 \end{bmatrix} \end{cases}$$

Since the second eigenvalue is positive, the critical point at $(1,0)$ is unstable.

  - Direction field plot



(b)

$$\frac{dx}{dt} = x(1 - 0.5y)$$
$$\frac{dy}{dt} = y(-0.25 + 0.5x)$$

- We have critical points at

$$(0,0) \quad , \quad (0.5, 2)$$

  and

$$\frac{\mathbf{dF}}{\mathbf{dx}} = \begin{bmatrix} 1 - 0.5y & -0.5x \\ 0.5x & -0.25 + 0.5x \end{bmatrix}$$

  - Linearized system at $(0,0)$:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & -0.25 \end{bmatrix}$$
$$\Longrightarrow \quad \mathbf{y}(t) = c_1 e^t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{-0.25t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The critical point at $(0,0)$ is thus unstable since the first eigenvalue $r_1 = 1$ is positive.

– Linearized system at $(0.5, 2)$:

$$\mathbf{A} = \begin{bmatrix} 0 & -0.25 \\ 0.25 & 0 \end{bmatrix}$$

$$\text{eigenvalues/eigenvectors} \quad \rightarrow \quad \begin{cases} r_1 = 0.25i \quad , \quad \xi_1 = \begin{bmatrix} 1 \\ -i \end{bmatrix} \\ r_2 = -0.25i \quad , \quad \xi_2 = \begin{bmatrix} 1 \\ i \end{bmatrix} \end{cases}$$

$$\begin{aligned} \mathbf{y}(t) &= c_1 e^{0.25it} \begin{bmatrix} 1 \\ -i \end{bmatrix} + c_2 e^{-0.25it} \begin{bmatrix} 1 \\ i \end{bmatrix} \\ &= c_1 \begin{bmatrix} \cos(t/4) + i\sin(t/4) \\ -i\cos(t/4) + \sin(t/4) \end{bmatrix} + c_2 \begin{bmatrix} \cos(t/4) - i\sin(t/4) \\ i(\cos(t/4) - i\sin(t/4)) \end{bmatrix} \\ &= c_1 \begin{bmatrix} \cos(t/4) \\ \sin(t/4) \end{bmatrix} + c_2 \begin{bmatrix} \sin(t/4) \\ -\cos(t/2) \end{bmatrix} \end{aligned}$$

Since the eigenvalues are both pure imaginary, analysis of this critical point by linearization is inconclusive.

– Direction field plot