Math 4233
Solutions to Homework Set 2

Before applying various numerical methods, let's write down the exact solution of

$$\begin{aligned} x' &= 2x - 3t \\ x(0) &= 1 \end{aligned}$$

This is a first order, linear, non-homogeneous ODE with an initial condition. Such ODE/BVPs can be solved exactly as follows.

$$x' + p(t)\,x = g(t) \quad , \quad x(0) = x_0$$

$$\implies \quad x(t) = \frac{1}{\mu_0(t)} \int_0^t \mu_0(s)\,g(s)\,ds + \frac{x_0}{\mu_0(t)} \qquad \text{where } \mu_0(t) := \exp\left(\int_0^t p(s)\,ds\right)$$

In the case at hand,

$$p(t) = -2 \quad , \quad g(t) = -3t \quad , \quad x_0 = 1$$

and so we have

$$\begin{aligned} \mu_0(t) &= \exp\left(\int_0^t -2ds\right) = \exp\left(-2s|_0^t\right) = e^{-2t} \\ x(t) &= \frac{1}{e^{-2t}} \int_0^t e^{-2s}(-3s)\,ds + \frac{1}{e^{-2t}} \\ &= e^{2t}\left(\frac{3}{2}te^{-2t} + \frac{3}{4}e^{-2t} - \frac{3}{4}\right) + e^{2t} \\ &= \frac{3}{2}t + \frac{3}{4} + \frac{1}{4}e^{2t} \end{aligned}$$

The "exact" value of the solution at $t = 0.4$ is thus

$$x(0.4) = 1.906385232$$

1. Use the Euler method with a step size of 0.1 to determine an approximate value of the solution of

$$x' = 2x - 3t \quad , \quad x(0) = 1 \tag{1}$$

at $t = 0.4$.. Do the same using a step size of 0.01.

- The initial values $t_0 = 0$, $x_0 := x(t_0) = 1$, $h = 0.1$, $F(t,x) := 2x - 3t$, and the recursive algorithm

  $$\begin{aligned} t_i &= t_{i-1} + h \\ x_i &= x_{i-1} + F(t_{i-1}, x_{i-1})\,h \end{aligned}$$

  yield the following table

$t_0 = 0.0$	$x_0 = 1$
$t_1 = 0.1$	$x_1 = 1.20$
$t_2 = 0.2$	$x_2 = 1.410$
$t_3 = 0.3$	$x_3 = 1.6320$
$t_4 = 0.4$	$x_4 = 1.86840$

  So we conclude

  $$x(0.4) = x(t_4) \approx x_4 = 1.86840$$

  The percentage error in this case is

  $$error = \frac{x_{exact}(0.4) - x_4}{x_{exact}(0.t)} = 1.992527\%$$

- Repeating the computation using $h = 0.01$ (now requiring 40 steps), we obtain

$$x(0.4) = x(t_{40}) \approx x_{40} = 1.902009916$$

The percentage error in this case is

$$error = \frac{x_{exact}(0.4) - x_{40}}{x_{exact}(0.t)} = -.229508\%$$

2. Use the Huen method (also known as the second order Runge-Kutta method) with a step-size of 0.01 to compute an approximate value for the solution of (1) at $t = 0.4$.

- The Huen method corresponds to the following algorithm:

$$\begin{aligned} t_i &= t_{i-1} + h \\ x_i &= x_{i-1} + \frac{1}{2}\left(F(t_{i-1}, x_{i-1}) + F(t_{i-1} + h, x_{i-1} + F(t_{i-1}, x_{i-1})\, h)\right) h \end{aligned}$$

or, inserting a couple intermediary calculations as a substeps,

$$\begin{aligned} t_i &= t_{i-1} + h \\ f_1 &= F(t_{i-1}, x_{i-1}) \\ f_2 &= F(t_{i-1} + h, x_{i-1} + hf_1) \\ x_i &= x_{i-1} + \frac{h}{2}(f_1 + f_2) \end{aligned}$$

This algorithm yields a table like

$$\begin{array}{ll} t_0 = 0.0 & x_0 = 1 \\ t_1 = 0.01 & x_1 = 1.020050000 \\ t_2 = 0.02 & x_2 = 1.040202010 \\ t_3 = 0.03 & x_3 = 1.060458091 \\ \vdots & \vdots \\ t_{39} = 0.39 & x_{39} = 1.880340130 \\ t_{40} = 0.40 & x_{40} = 1.906356001 \end{array}$$

We conclude

$$x(0.4) = x(t_{40}) \approx x_{40} = 1.906356001$$

The percentage error in this case is

$$\%\ error = \frac{x_{exact}(0.4) - x_{40}}{x_{exact}(0.t)} = -.001533\%$$

3. Use the fourth order Runge-Kutta method with a step-size of 0.01 to compute an approximate value for the solution of (1) at $t = 0.4$.

- The iterative computations for the fourth order Runge-Kutta method are given by

$$\begin{aligned} t_i &= t_{i-1} + h \\ f_{1,i-1} &= F(t_{i-1}, x_{i-1}) \\ f_{2,i-1} &= F(t_{i-1} + h/2, x_{i-1} + hF_{1,i-1}/2) \\ f_{3,i-1} &= F(t_{i-1} + h/2, x_{i-1} + hF_{2,i-1}/2) \\ f_{4,i-1} &= F(t_{i-1} + h, x_{i-1} + hF_{3,i-1}) \\ x_i &= x_{i-1} + \frac{h}{6}(f_{1,i-1} + 2f_{2,i-1} + 2f_{3,i-1} + f_{4,i-1}) \end{aligned}$$

This leads to a table of the form

$$\begin{array}{ll} t_0 = 0.0 & x_0 = 1 \\ t_1 = 0.01 & x_1 = 1.020050335 \\ t_2 = 0.02 & x_2 = 1.040202694 \\ t_3 = 0.03 & x_3 = 1.060459137 \\ t_4 = 0.04 & x_4 = 1.080821767 \\ t_5 = 0.05 & x_5 = 1.101292730 \\ \vdots & \vdots \\ t_{38} = 0.38 & x_{38} = 1.854569056 \\ t_{39} = 0.39 & x_{39} = 1.880368067 \\ t_{40} = 0.40 & x_{40} = 1.906385233 \end{array}$$

The percentage error in this case is

$$\%\ error = \frac{x_{exact}(0.4) - x_{40}}{x_{exact}(0.t)} = 0.00000\%$$

4. Use the fourth order Adams-Bashforth multi-step method with a step-size of 0.01 to compute an approximate value for the solution of (1) at $t = 0.4$.

- The fourth order Adams-Bashforth uses the following recursive formula.

$$t_i = t_{i-1} + h$$

$$x_i = x_{i-1} + \frac{h}{24}\left(55F(t_{i-1}, x_{i-1}) - 59F(t_{i-2}, x_{i-2}) + 37F(t_{i-3}, x_{i-3}) - 9F(t_{i-4}, x_{i-4})\right)$$

To implement this we need four initial points. These we'll take from the fourth order Runge-Kutta computation performed above. One obtains in this way a table of the form

$$\begin{array}{ll} t_0 = 0.0 & x_0 = 1 \\ t_1 = 0.01 & x_1 = 1.020050335 \\ t_2 = 0.02 & x_2 = 1.040202694 \\ t_3 = 0.03 & x_3 = 1.060459137 \\ t_4 = 0.04 & x_4 = 1.080821767 \\ t_5 = 0.05 & x_5 = 1.101292729 \\ \vdots & \vdots \\ t_{38} = 0.38 & x_{38} = 1.854569037 \\ t_{39} = 0.39 & x_{39} = 1.880368047 \\ t_{40} = 0.40 & x_{40} = 1.906385212 \end{array}$$

The percentage error in this case is

$$\%\ error = \frac{x_{exact}(0.4) - x_{40}}{x_{exact}(0.t)} = -.000001\%$$