Math 4233
Solution to Homework Set 1

1. Find the inverses of the following matrices:

(a) $\begin{pmatrix} 1 & 4 \\ -2 & 3 \end{pmatrix}$

- For invertible 2×2 matrices the following identity (easily derived from the cofactor expression for $\mathbf{A}^{-1}$) holds

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

and so

$$\begin{pmatrix} 1 & 4 \\ -2 & 3 \end{pmatrix}^{-1} = \frac{1}{(1)(3) - (4)(-2)} \begin{pmatrix} 3 & -4 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} \frac{3}{11} & -\frac{4}{11} \\ \frac{2}{11} & \frac{1}{11} \end{pmatrix}$$

(b) $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}$

- Row reduction is usually the fastest way to determine the inverse of a matrix of rank > 2.

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 4 & 5 & 0 & 1 & 0 \\ 3 & 5 & 6 & 0 & 0 & 1 \end{array}\right) \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - 2R_1} \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 0 & -1 & -2 & 1 & 0 \\ 0 & -1 & -3 & -3 & 0 & 1 \end{array}\right) \xrightarrow{R_2 \longleftrightarrow R_3} \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -3 & -3 & 0 & 01 \\ 0 & 0 & -1 & -2 & 1 & 0 \end{array}\right)$$

$$\xrightarrow[\substack{R_2 \to -R_2 \\ R_3 \to -R_3}]{R_1 \longleftrightarrow R_1 + 2R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & -3 & -5 & 0 & 2 \\ 0 & 1 & 3 & 3 & 0 & -1 \\ 0 & 0 & 1 & 2 & -1 & 0 \end{array}\right) \xrightarrow[R_2 \to R_2 - 3R_3]{R_1 \longleftrightarrow R_1 + 3R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -3 & 2 \\ 0 & 1 & 0 & -3 & 3 & -1 \\ 0 & 0 & 1 & 2 & -1 & 0 \end{array}\right)$$

So

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{pmatrix}$$

(c) $\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$

- The formula used in 2(a) works also for complex matrices, and so we have

$$\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}^{-1} = \frac{1}{(1)(1) - (i)(i)} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} = \frac{1}{1+1} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{i}{2} \\ -\frac{i}{2} & \frac{1}{2} \end{pmatrix}$$

2. Find the eigenvectors and eigenvalues of the following matrices:

(a) $\mathbf{A} = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix}$

- The characteristic polynomial is

$$\det \begin{pmatrix} 5-\lambda & -1 \\ 3 & 1-\lambda \end{pmatrix} = (5-\lambda)(1-\lambda) + 3 = \lambda^2 - 6\lambda + 8 = (\lambda - 2)(\lambda - 4)$$

The roots of the characteristic polynomial are thus $\lambda = 2, 4$. These are the eigenvalues of $\mathbf{A}$. To find the corresponding eigenvectors we find a basis for the null spaces of $\mathbf{A} - \lambda\mathbf{I}$ for $\lambda = 2, 4$:

$\lambda = 2:$

$$NullSp(\mathbf{A}-2\mathbf{I}) = NullSp\begin{pmatrix} 5-2 & -1 \\ 3 & 1-2 \end{pmatrix} = NullSp\begin{pmatrix} 3 & -1 \\ 3 & -1 \end{pmatrix} = NullSp\begin{pmatrix} 3 & -1 \\ 0 & 0 \end{pmatrix}$$

And so if $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ belongs to the null space of $(\mathbf{A}-2\mathbf{I})$ it must satisfy

$$3x_1 - x_2 = 0 \quad \Longrightarrow \quad x_2 = 3x_1 \quad \Longrightarrow \quad \mathbf{x} = \begin{bmatrix} x_1 \\ 3x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

and so $\mathbf{v}_{\lambda=2} := \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ will be an eigenvector of $\mathbf{A}$ with eigenvalue 2.

$\lambda = 4:$

$$\begin{aligned} NullSp(\mathbf{A}-4\mathbf{I}) &= NullSp\begin{pmatrix} 5-4 & -1 \\ 3 & 1-4 \end{pmatrix} = NullSp\begin{pmatrix} 1 & -1 \\ 3 & -2 \end{pmatrix} = NullSp\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \\ &\Longrightarrow \quad x_1 - x_2 = 0 \quad \Longrightarrow \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{aligned}$$

and so $\mathbf{v}_{\lambda=4} := \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ will be an eigenvector of $\mathbf{A}$ with eigenvalue 4.

(b) $\mathbf{A} = \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$

- We calculate as in Problem 2(a), only using complex arithmetic.

$$0 = \det\begin{pmatrix} 1-\lambda & i \\ i & 1-\lambda \end{pmatrix} = (1-\lambda)^2 + 1 = \lambda^2 - 2\lambda + 2 \quad \Longrightarrow \quad \lambda = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(2)}}{2(1)} = \frac{2 \pm \sqrt{-4}}{2} = 1 \pm i$$

$\lambda = 1+i:$

$$\Longrightarrow \quad NullSp\begin{pmatrix} 1-(1+i) & i \\ i & 1-(1+i) \end{pmatrix} = NullSp\begin{pmatrix} -i & i \\ i & -i \end{pmatrix} = NullSp\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$$

$$\mathbf{x} \in NullSp\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \quad \Longrightarrow \quad x_1 - x_2 = 0 \quad \Longrightarrow \quad \mathbf{x} = x_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\lambda = 1-i:$

$$\Longrightarrow \quad NullSp\begin{pmatrix} 1-(1-i) & i \\ i & 1-(1-i) \end{pmatrix} = NullSp\begin{pmatrix} i & i \\ i & i \end{pmatrix} = NullSp\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\mathbf{x} \in NullSp\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \quad \Longrightarrow \quad x_1 + x_2 = 0 \quad \Longrightarrow \quad \mathbf{x} = x_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

We thus find the following eigenvalue/eigenvector pairs

$$\begin{aligned} \lambda &= 1+i \quad , \quad \mathbf{v}_{\lambda=1+i} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ \lambda &= 1-i \quad , \quad \mathbf{v}_{\lambda=1-i} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \end{aligned}$$

(c) $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{pmatrix}$, eigenvalues: $1, 1+2i, 1-2i$

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$$\det(\mathbf{A}-\lambda\mathbf{I}) = \det\begin{pmatrix} 1-\lambda & 0 & 0 \\ 2 & 1-\lambda & -2 \\ 3 & 2 & 1-\lambda \end{pmatrix} = (1-\lambda)((1-\lambda)(1-\lambda) - (-2)(2)) = (1-\lambda)\left(\lambda^2 - 2\lambda + 5\right)$$

One root of the characteristic polynomial is obvious: $\lambda = 1$. To get the other roots, we apply the quadratic formula to the second factor

$$0 = \lambda^2 - 2\lambda + 5 \quad \Longrightarrow \quad \lambda = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(5)}}{2(1)} = \frac{2 \pm \sqrt{-16}}{2} = 1 \pm 2i$$

We thus have three eigenvalues $\lambda = 1,\ 1+2i,\ 1-2i$

$\lambda = 1:$

$$NullSp(\mathbf{A} - \mathbf{I}) = NullSp\begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & -2 \\ 3 & 2 & 0 \end{pmatrix} = NullSp\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{x} \in NullSp(\mathbf{A} - \mathbf{I}) \Longrightarrow \begin{cases} x_1 - x_3 = 0 \\ x_2 + \frac{3}{2}x_3 = 0 \end{cases} \Longrightarrow \mathbf{x} = x_3 \begin{bmatrix} 1 \\ -\frac{3}{2} \\ 1 \end{bmatrix}$$

$\lambda = 1 + 2i$

$$NullSp(\mathbf{A} - (1+2i)\mathbf{I}) = NullSp\begin{pmatrix} -2i & 0 & 0 \\ 2 & -2i & -2 \\ 3 & 2 & -2i \end{pmatrix} = NullSp\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -i \\ 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{x} \in NullSp(\mathbf{A} - (1+2i)\mathbf{I}) \Longrightarrow \begin{cases} x_1 = 0 \\ x_2 - ix_3 = 0 \end{cases} \Longrightarrow \mathbf{x} = x_2 \begin{bmatrix} 0 \\ 1 \\ -i \end{bmatrix}$$

$\lambda = 1 - 2i$

$$NullSp(\mathbf{A} - (1+2i)\mathbf{I}) = NullSp\begin{pmatrix} 2i & 0 & 0 \\ 2 & 2i & -2 \\ 3 & 2 & 2i \end{pmatrix} = NullSp\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & i \\ 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{x} \in NullSp(\mathbf{A} - (1+2i)\mathbf{I}) \Longrightarrow \begin{cases} x_1 = 0 \\ x_2 + ix_3 = 0 \end{cases} \Longrightarrow \mathbf{x} = x_2 \begin{bmatrix} 0 \\ 1 \\ i \end{bmatrix}$$

3. For each of the matrices $\mathbf{A}$ in Problem 2 find a diagonal matrix $\mathbf{D}$ and an invertible matrix $\mathbf{C}$ such that

$$\mathbf{D} = \mathbf{C}^{-1}\mathbf{A}\mathbf{C}$$

- The diagonal matrix $\mathbf{D}$ is formed by arranging the eigenvalues of $\mathbf{A}$ as the entries of a diagonal matrix. The matrix $\mathbf{C}$ that converts $\mathbf{A}$ to the diagonal matrix $\mathbf{D}$ is formed by using the eigenvectors of $\mathbf{A}$ as columns - ordered in such a way that the i^{th} column vector of $\mathbf{C}$ is the eigenvector corresponding to the eigenvalue used as the i^{th} diagonal entry of $\mathbf{D}$.

(a)

$$\mathbf{A} = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix} \Longleftrightarrow \mathbf{D} = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix} \quad , \quad \mathbf{C} = \begin{pmatrix} 1 & 1 \\ 3 & 1 \end{pmatrix}$$

(b)

$$\mathbf{A} = \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \Longleftrightarrow \mathbf{D} = \begin{pmatrix} 1+i & 0 \\ 0 & 1-i \end{pmatrix} \quad , \quad \mathbf{C} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

(c)

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{pmatrix} \Longleftrightarrow \mathbf{D} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1+2i & 0 \\ 0 & 0 & 1-2i \end{pmatrix} \quad , \quad \mathbf{C} = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{3}{2} & 1 & 1 \\ 1 & -i & i \end{pmatrix}$$

4. For each of the following systems find the fundamental (independent) solutions.

(a) $\frac{d\mathbf{x}}{dt} = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \mathbf{x}$

- $$0 = \det\left(\begin{bmatrix} 3-\lambda & -2 \\ 2 & -2-\lambda \end{bmatrix}\right) = \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1) \quad \Longrightarrow \quad \lambda = 2, 1$$

  eigenvector for $\lambda = 2$:

  $$NullSp\left(\begin{bmatrix} 3-2 & -2 \\ 2 & -2-2 \end{bmatrix}\right) = NullSp\left(\begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}\right) \quad \Longrightarrow \quad \begin{cases} v_1 = 2v_2 \\ 0 = 0 \end{cases}$$

  $$\Longrightarrow \quad \mathbf{v}_{\lambda=2} = v_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} \sim \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

  eigenvector for $\lambda = -1$:

  $$NullSp\left(\begin{bmatrix} 3-(-1) & -2 \\ 2 & -2-(-1) \end{bmatrix}\right) = NullSp\left(\begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}\right) \quad \Longrightarrow \quad \begin{cases} v_2 = 2v_1 \\ 0 = 0 \end{cases}$$

  $$\Longrightarrow \quad \mathbf{v}_{\lambda=-1} = v_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} \sim \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

  The fundamental solution corresponding to an eigevalue λ_i with eigenvector $\mathbf{v}_i$ is $\mathbf{x}^{(i)} = e^{\lambda_i t}\mathbf{v}_i$. So we have

  $$\mathbf{x}^{(1)}(t) = e^{2t}\begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad , \qquad \mathbf{x}^{(2)}(t) = e^{-t}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

(b) $\frac{d\mathbf{x}}{dt} = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} \mathbf{x}$

We have

$$\det\left(\begin{bmatrix} 2-\lambda & -5 \\ 1 & -2-\lambda \end{bmatrix}\right) = \lambda^2 + 1 \quad \Longrightarrow \quad \lambda = \pm i$$

eigenvector for $\lambda = i$:

$$NullSp\left(\begin{bmatrix} 2-i & -5 \\ 1 & -2-i \end{bmatrix}\right) = NullSp\begin{pmatrix} 1 & -2-i \end{pmatrix} \quad \Longrightarrow \quad v_1 = (2+i)\, v_2$$

$$\Longrightarrow \quad \mathbf{v}_{\lambda=i} = v_1 \begin{bmatrix} 2+i \\ 1 \end{bmatrix} \sim \begin{bmatrix} 2+i \\ 1 \end{bmatrix}$$

The eigenvector for $\lambda = -i$ will simply be the complex conjugate of the eigenvector for $\lambda = i$:proof: if $\mathbf{A}$ is a real matrix and $\mathbf{Av} = \lambda\mathbf{v}$, then

$$\mathbf{Av} = \lambda\mathbf{v} \quad \Longrightarrow \quad \overline{(\mathbf{Av})} = \overline{(\lambda\mathbf{v})} \quad \Longrightarrow \quad \mathbf{A}\overline{\mathbf{v}} = \overline{\lambda}\overline{\mathbf{v}} \quad \Longrightarrow \quad \overline{\mathbf{v}} \text{ is an eigenvector for } \mathbf{A} \text{ with eigenvalue } \overline{\lambda}$$

Hence,

$$\mathbf{v}_{\lambda=-i} = \overline{\mathbf{v}_{\lambda=i}} = \begin{bmatrix} 2-i \\ 1 \end{bmatrix}$$

Corresponding to the eigenvalue/eigenvector pairs we have the following two (complex) fundamental solutions

$$\mathbf{x}_{\mathbb{C}}^{(1)}(t) = e^{it}\begin{bmatrix} 2+i \\ 1 \end{bmatrix} = (\cos t + i\sin t)\begin{bmatrix} 2+i \\ 1 \end{bmatrix} = \begin{bmatrix} 2\cos t - \sin t + i(\cos t + 2\sin t) \\ \cos t + i\sin t \end{bmatrix}$$

$$\mathbf{x}_{\mathbb{C}}^{(2)}(t) = e^{-it}\begin{bmatrix} 2-i \\ 1 \end{bmatrix} = (\cos t - i\sin t)\begin{bmatrix} 2-i \\ 1 \end{bmatrix} = \begin{bmatrix} 2\cos t - \sin t + i(-\cos t - 2\sin t) \\ \cos t - i\sin t \end{bmatrix}$$

Notice that $\overline{\mathbf{x}_{\mathbb{C}}^{(1)}(t)} = \mathbf{x}_{\mathbb{C}}^{(2)}(t)$. To get two real-valued fundamental solutions we can simply use the real and imaginary parts of $\mathbf{x}_{\mathbb{C}}^{(1)}(t)$ (or $\mathbf{x}_{\mathbb{C}}^{(2)}(t)$)

$$\Longrightarrow \quad \mathbf{x}^{(1)}(t) = \begin{bmatrix} 2\cos t - \sin t \\ \cos t \end{bmatrix} \qquad , \qquad \mathbf{x}^{(2)}(t) = \begin{bmatrix} \cos t + 2\sin t \\ \sin t \end{bmatrix}$$

(c) $\frac{d\mathbf{x}}{dt} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -8 & -5 & -3 \end{bmatrix} \mathbf{x}$, eigenvectors: $\left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\} \leftrightarrow 2, \left\{ \begin{bmatrix} 1 \\ -\frac{5}{4} \\ -\frac{7}{4} \end{bmatrix} \right\} \leftrightarrow -2, \left\{ \begin{bmatrix} 1 \\ -\frac{4}{3} \\ -\frac{2}{3} \end{bmatrix} \right\} \leftrightarrow -1$

$$0 = \det\left(\begin{bmatrix} 1-\lambda & 1 & 1 \\ 2 & 1-\lambda & -1 \\ -8 & -5 & -3-\lambda \end{bmatrix} \right) = -\lambda^3 - \lambda^2 + 4\lambda + 4 = -(\lambda - 2)(\lambda + 2)(\lambda + 1)$$
$$\Longrightarrow \quad \lambda = 2, -2, 1$$

$$NullSp(\mathbf{A} - 2\mathbf{I}) = NullSp\left(\begin{bmatrix} 1-2 & 1 & 1 \\ 2 & 1-2 & -1 \\ -8 & -5 & -3-2 \end{bmatrix} \right) = NullSp\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \right)$$
$$\Longrightarrow \quad \left\{ \begin{array}{c} v_1 = 0 \\ v_3 = -v_2 \end{array} \right. \quad \Longrightarrow \quad \mathbf{v}_{\lambda=2} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

$$NullSp(\mathbf{A} - (-2)\mathbf{I}) = NullSp\left(\begin{bmatrix} 1+2 & 1 & 1 \\ 2 & 1+2 & -1 \\ -8 & -5 & -3+2 \end{bmatrix} \right) = NullSp\left(\begin{bmatrix} 1 & 0 & \frac{4}{7} \\ 0 & 1 & -\frac{5}{7} \\ 0 & 0 & 0 \end{bmatrix} \right)$$
$$\Longrightarrow \quad \left\{ \begin{array}{c} v_1 = -\frac{4}{7} v_3 \\ v_2 = \frac{5}{7} v_3 \end{array} \right. \quad \Longrightarrow \quad \mathbf{v}_{\lambda=-2} = \begin{bmatrix} -\frac{4}{7} \\ \frac{5}{7} \\ 1 \end{bmatrix}$$

,

$$NullSp(\mathbf{A} - (-1)\mathbf{I}) = NullSp\left(\begin{bmatrix} 1+1 & 1 & 1 \\ 2 & 1+1 & -1 \\ -8 & -5 & -3+1 \end{bmatrix} \right) = NullSp\left(\begin{bmatrix} 1 & 0 & \frac{3}{2} \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \right)$$
$$\Longrightarrow \quad \left\{ \begin{array}{c} v_1 = -\frac{3}{2} v_3 \\ v_2 = 2 v_3 \end{array} \right. \quad \Longrightarrow \quad \mathbf{v}_{\lambda=-1} = \begin{bmatrix} -\frac{3}{2} \\ 2 \\ 1 \end{bmatrix}$$

We thus have the following three fundamental solutions

$$\mathbf{x}^{(1)}(t) = e^{2t} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \quad , \quad \mathbf{x}^{(2)}(t) = e^{-2t} \begin{bmatrix} -\frac{4}{7} \\ \frac{5}{7} \\ 1 \end{bmatrix} \quad , \quad \mathbf{x}^{(3)}(t) = e^{-t} \begin{bmatrix} -\frac{3}{2} \\ 2 \\ 1 \end{bmatrix}$$

5. Find the general solution of the following systems:

(a) $\frac{d\mathbf{x}}{dt} = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} \mathbf{x}$

$$0 = \det\left(\begin{bmatrix} 3-\lambda & -4 \\ 1 & -1-\lambda \end{bmatrix} \right) = \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2 \quad \Longrightarrow \quad \lambda = 1$$

$$NullSp\left(\begin{bmatrix} 3-1 & -4 \\ 1 & -1-1 \end{bmatrix} \right) = NullSp\begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix} \quad \Longrightarrow \quad \{ v_1 = 2v_2$$
$$\Longrightarrow \quad \mathbf{v}_{\lambda=1} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Note that we have only one eigenvalue with one eigenvector. This leads to the fundamental solution

$$\mathbf{x}^{(1)}(t) = e^{t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

but we still need a second independent solution.

Following the method of §7.8, we find a second fundamental solution as follows. If

$$\mathbf{x}^{(1)}(t) = e^{\lambda t}\boldsymbol{\xi}$$

is the first fundamental solution (with corresponding eigenvalue λ and corresponding eigenvector $\boldsymbol{\xi}$), we first solve the linear system $(\mathbf{A} - \lambda\mathbf{I})\boldsymbol{\eta} = \boldsymbol{\xi}$ for (the generalized eigenvector) $\boldsymbol{\eta}$. A second independent solution of the original system will then be

$$\mathbf{x}^{(2)}(t) = te^{\lambda t}\boldsymbol{\xi} + e^{\lambda t}\boldsymbol{\eta}$$

Applying this procedure to the case at hand:

$$\begin{bmatrix} 3-1 & -4 \\ 1 & -1-1 \end{bmatrix}\begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \Longrightarrow \quad \eta_1 - 2\eta_2 = 1$$

$$\Longrightarrow \quad \boldsymbol{\eta} = \begin{bmatrix} 1+2\eta_2 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \eta_2\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Or second solution will thus be

$$\mathbf{x}^{(2)}(t) = te^t\begin{bmatrix} 2 \\ 1 \end{bmatrix} + e^t\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \eta_2\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right)$$

Notice that the piece

$$\eta_2 e^t\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

is just a scalar multiple of $\mathbf{x}^{(1)}(t)$ and isn't really essential to the linear independence of the solution $\mathbf{x}^{(2)}(t)$ from the solution $\mathbf{x}^{(1)}(t)$. We can just drop it (e.g., by choosing $\eta_2 = 0$). We thus have

$$\mathbf{x}^{(1)}(t) = e^t\begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad , \quad \mathbf{x}^{(2)}(t) = te^t\begin{bmatrix} 2 \\ 1 \end{bmatrix} + e^t\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

as two fundamental solutions. The general solution can be written as

$$\mathbf{x}(t) = c_1\mathbf{x}^{(1)}(t) + c_2\mathbf{x}^{(2)}(t) = c_1\begin{bmatrix} 2e^t \\ e^t \end{bmatrix} + c_2\begin{bmatrix} 2te^t + e^t \\ te^t \end{bmatrix}$$

or

$$\mathbf{x}(t) = \begin{bmatrix} 2e^t & 2te^t + e^t \\ e^t & te^t \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

(b) $\frac{d\mathbf{x}}{dt} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}\mathbf{x}$

We have

$$\det\left(\begin{bmatrix} 1-\lambda & 1 & 1 \\ 2 & 1-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{bmatrix}\right) = -\lambda^3 + 3\lambda^2 - 4 = -(1+\lambda)(\lambda-2)^2 \quad \Longrightarrow \quad \lambda = -1, 2$$

The eigenvector corresponding the eigenvalue $\lambda = -1$ is easily seen to be

$$\mathbf{v}_{\lambda=-1} = \begin{bmatrix} -\frac{3}{2} \\ 2 \\ 1 \end{bmatrix}$$

And the eigenvector corresponding to the eigenvalue $\lambda = 2$

$$\mathbf{v}_{\lambda=2} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

For each of these eigenvalue/eigenvector pairs we have a fundamental solution

$$\mathbf{x}^{(1)}(t) = e^{-t}\begin{bmatrix} -\frac{3}{2} \\ 2 \\ 1 \end{bmatrix} \quad , \quad \mathbf{x}^{(2)}(t) = e^{2t}\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

However, because we are dealing with a 3×3 linear system, there must be a third independent solution. Note that the characteristic polynomial of the coefficient matrix $\mathbf{A}$ has two factors of $(\lambda - 2)$, and yet $\mathbf{A}$ has only one eigenvector with eigenvalue 2. This means that $\mathbf{A}$ must have another *generalized eigenvector* with eigenvalue 2. We find it by solving $(\mathbf{A} - 2\mathbf{I})\,\boldsymbol{\eta} = \boldsymbol{\xi}$ where ξ is the eigenvector of $\mathbf{A}$ corresponding to $\lambda = 2$. Thus we need to solve

$$\begin{bmatrix} 1-2 & 1 & 1 \\ 2 & 1-2 & -1 \\ 0 & -1 & 1-2 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \quad \Longrightarrow \quad \begin{cases} \eta_1 = -1 \\ \eta_2 = -1 - \eta_3 \end{cases}$$

$$\Longrightarrow \quad \boldsymbol{\eta} = \begin{bmatrix} 1 \\ 1-\eta_3 \\ \eta_3 \end{bmatrix}$$

We can now write down a third independent solution

$$\mathbf{x}^{(t)}(t) = te^{2t}\mathbf{v}_{\lambda=2} + e^{2t}\boldsymbol{\eta} = te^{2t}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + e^{2t}\begin{bmatrix} 1 \\ 1-\eta_3 \\ \eta_3 \end{bmatrix} = te^{2t}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + e^{2t}\begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} + \eta_3 e^{2t}\begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix}$$

$$= te^{2t}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + e^{2t}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

where again we dropped the last term since it is just a scalar multiple of the second solution $\mathbf{x}^{(2)}(t)$.

The general solution is thus

$$\begin{aligned} \mathbf{x}(t) &= c_1\mathbf{x}^{(1)}(t) + c_2\mathbf{x}^{(2)}(t) + c_3\mathbf{x}^{(3)}(t) \\ &= c_1\begin{bmatrix} -\frac{3}{2}e^{-t} \\ 2e^{-t} \\ e^{-t} \end{bmatrix} + c_2\begin{bmatrix} 0 \\ -e^{2t} \\ e^{2t} \end{bmatrix} + c_3\begin{bmatrix} -e^{2t} \\ (-1+t)\,e^{2t} \\ -te^{2t} \end{bmatrix} \\ &= \begin{bmatrix} -\frac{3}{2}e^{-t} & 0 & -e^{2t} \\ 2e^{-t} & -e^{2t} & (-1+t)\,e^{2t} \\ e^{-t} & e^{2t} & -te^{2t} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \end{aligned}$$

And so the fundamental matrix is

$$\boldsymbol{\Phi}(t) = \begin{bmatrix} -\frac{3}{2}e^{-t} & 0 & -e^{2t} \\ 2e^{-t} & -e^{2t} & (-1+t)\,e^{2t} \\ e^{-t} & e^{2t} & -te^{2t} \end{bmatrix}$$