Math 4233

SOLUTIONS TO SECOND EXAM

Thursday, July 19, 2012

1. Find the solution of the following heat conduction problem. Explain the steps you take in solving this problem in as much detail as possible.

(1a) $$4u_t - u_{xx} = 0 \quad , \quad 0 < x < 2 \quad , \quad t > 0$$

(1b) $$u(0,t) = 0$$

(1c) $$u(2,t) = 0$$

(1d) $$u(x,0) = 4\sin(6\pi x)$$

- We first use separation of variables to find a suitable family of solutions of the heat equation satisfying the first two boundary conditions. Thus, we look for functions of the form

(1e) $$u(x,t) = X(x)\,Y(t)$$

that will satisfy (1a), (1b) and (1c). Substituting (1e) into (1a) and dividing the result by $X(x)\,Y(t)$ we obtain

$$4\frac{\dot{Y}}{Y} = \frac{X''}{X}$$

Since the left hand side does not depend on x and the right hand side does not depend on t, we conclude that both sides must be equal to a constant, which we shall denote by $-\lambda^2$. We are thus led to

$$X'' = -\lambda^2 X \quad \Longrightarrow \quad X(x) = A\sin(\lambda x + \delta)$$

$$\dot{Y} = -\frac{\lambda^2}{4}Y \quad \Longrightarrow \quad Y(t) = Ce^{-\frac{\lambda^2}{4}t}$$

Imposing the boundary conditions at $x = 0$ on the expression (1e) we find

$$0 = u(0,t) = ACe^{-\frac{\lambda^2}{4}t}\sin(\delta) \quad \Longrightarrow \quad \delta = 0 \qquad \text{(for non-trivial solutions)}$$

Taking then $\delta = 0$ and imposing the boundary condition at $x = 2$ we find

$$0 = u(2,t) = ACe^{-\frac{\lambda^2}{4}t}\sin(2\lambda) \quad \Longrightarrow \quad \lambda = \frac{n\pi}{2} \quad , \quad n = 1,2,\ldots \qquad \text{(for non-trivial solutions)}$$

Thus, any function of the form

$$\phi_n(x,t) = e^{-\frac{1}{4}\left(\frac{n\pi}{2}\right)^2 t}\sin\left(\frac{n\pi}{2}x\right) = e^{-\left(\frac{n\pi}{4}\right)^2 t}\sin\left(\frac{n\pi}{2}x\right) \qquad , \qquad n = 1,2,\ldots$$

will satisfy equations (1a) - (1c). Moreover, any linear combination of the functions ϕ_n will continue to satisfy equations (1a) - (1c). We thus set

(1f) $$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-\left(\frac{n\pi}{4}\right)^2 t}\sin\left(\frac{n\pi}{2}x\right) =$$

and try to choose the coefficients a_n so that the final boundary condition (1d) is satisfied. Plugging (1f) into (1d) we obtain

$$4\sin(6\pi x) = \sum_{n=1}^{\infty} a_n e^{-\left(\frac{n\pi}{4}\right)^2 0}\sin\left(\frac{n\pi}{2}x\right) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi}{2}x\right)$$

From this we conclude that the coefficients a_n should coincide with coefficients of $\sin\left(\frac{n\pi}{2}x\right)$ in the Fourier-sine expansion of the function on the right hand side on the interval $[0,2]$. Thus, we set

$$\begin{aligned} a_n &= \frac{2}{L}\int_0^L 4\sin(6\pi x)\sin\left(\frac{n\pi}{L}x\right)dx \\ &= 4\int_0^2 \sin(6\pi x)\sin\left(\frac{n\pi}{2}x\right)dx \end{aligned}$$

By the orthogonality properties of the Fourier-sine functions

$$\frac{2}{L}\int_0^L \sin\left(\frac{6\pi}{L}x\right)\sin\left(\frac{n\pi}{L}x\right)dx = \begin{cases} 1 & \text{if } n = 6 \\ 0 & \text{if } n \neq 6 \end{cases}$$

we find

$$a_n = \begin{cases} 4 & \text{if } n = 12 \\ 0 & \text{otherwise} \end{cases}$$

Thus,

$$u\left(x,t\right) = 4e^{-\left(\frac{4\pi}{4}\right)^2 t}\sin\left(2\pi x\right) = 4e^{-\pi^2 t}\sin\left(2\pi x\right)$$

2. Find a stable, time-independent solution of

(2a) $$4u_t - u_{xx} = 0 \quad , \quad 0 \leq t \quad , \quad 0 \leq x \leq 2$$

(2b) $$u(0,t) = 2$$

(2c) $$u(2,t) = -2$$

and explain how you would employ it to develop a solution of

(2d) $$4u_t - u_{xx} = 0 \quad , \quad 0 < x < 2 \quad , \quad t > 0$$

(2e) $$u(0,t) = 1$$

(2f) $$u(2,t) = -1$$

(2g) $$u(x,0) = g(x)$$

(You can use results obtained in Problem #1 in your answer.)

- Suppose $u(t,x) = u_{ss}(x)$ is a time independent solution of (2a), (2b) and (2c). Then since $\frac{\partial}{\partial t} u_{ss}(x) = 0$, we must have
$$\frac{d^2 u_{ss}}{dx^2} = 0 \quad , \quad u_{ss}(0) = 2 \quad , \quad u_{ss}(2) = -2$$
The differential equation implies that u_{ss} is a linear function of x
$$\frac{d^2 u_{ss}}{dx^2} = 0 \quad \Longrightarrow \quad u_{ss} = Ax + B$$
If we then impose the boundary conditions at $x = 0$ and $x = 2$, we see that constants A and B must be chosen so that

(2h) $$u_{ss} = -2x + 1 \quad .$$

This will be our steady state solution.

We now set

(2i) $$u(x,t) = u_{ss}(x) + \phi(x,t)$$

and regard $\phi(x,t)$ as representing the discrepancy between the actual solution $u(x,t)$ and the eventual, steady-state solution of (2d)–(2g).

Plugging (2i) into (2d) yields
$$\phi_t - \phi_{xx} = 0 \quad , \quad 0 < t \quad , \quad 0 \leq x \leq 2$$
(since $\frac{\partial}{\partial t} u_{ss} = 0 = \frac{\partial^2}{\partial x^2} u_{ss}$) and the boundary conditions (2e), (2f) and (2g) require
$$1 = u(0,t) = u_{ss}(0) + \phi(0,t) = 1 + \phi(0,t) \quad \Longrightarrow \quad \phi(0,t) = 0$$
$$-1 = u(1,t) = u_{ss}(1) + \phi(1,t) = -1 + \phi(1,t) \quad \Longrightarrow \quad \phi(1,t) = 0$$
$$g(x) = u(x,0) = u_{xx}(x) + \phi(x,0) = -2x + 1 + \phi(x,0) \quad \Longrightarrow \quad \phi(x,0) = g(x) + 2x - 1$$
We need $\phi(x,t)$ to satisfy

(2j) $$4\phi_t - \phi_{xx} = 0 \quad , \quad 0 < x < 2 \quad , \quad t > 0$$

(2k) $$\phi(0,t) = 0$$

(2l) $$\phi(2,t) = 0$$

(2m) $$\phi(x,0) = g(x) + 2x - 1$$

From the results of Problem 1 (up to Eq. (1f)), we now that any function of the form

(2n) $$\phi(x,t) = \sum_{n=1}^{\infty} c_n e^{-\left(\frac{n\pi}{2}\right)^2 t} \sin\left(\frac{n\pi}{2} x\right)$$

will satisfy equations (2i) – (2l). Imposing (2m) on (2n) yields
$$g(x) + 2x - 2 = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{2} x\right)$$

If we multiply both sides by $\sin\left(\frac{m\pi}{2}x\right)$ and integrate over the interval $0 \leq x \leq 2$ we get

$$\int_0^2 \left(g\left(x\right) + 2x - 1\right) \sin\left(\frac{n\pi}{2}x\right) dx = \sum_{n=1}^{\infty} c_n \int \sin\left(\frac{n\pi}{2}x\right) \sin\left(\frac{m\pi}{2}x\right) dx = \sum_{n=1}^{\infty} c_n \delta_{m,n} = c_m$$

Thus, if we set

$$c_n = \int_0^2 \left(g\left(x\right) + 2x - 1\right) \sin\left(\frac{n\pi}{2}x\right) dx$$

then

$$u\left(x,t\right) = -2x + 1 + \sum_{n=1}^{\infty} c_n e^{-\left(\frac{n\pi}{2}\right)^2 t} \sin\left(\frac{n\pi}{2}x\right)$$

will satisfy the original PDE/BVP.

3. Find the solution of Laplace's equation

$$u_{xx}+u_{yy}=0 \tag{3a}$$

satisfying the boundary conditions

$$u(x,0)=0 \quad , \quad u(x,b)=g(x) \tag{3b}$$

$$u(0,y)=0 \quad , \quad u(a,y)=0 \tag{3c}$$

- We set

$$u(x,y)=X(x)\,Y(y) \tag{3d}$$

and substitute (3d) into (3a) and then divide the resulting equation by $X(x)\,Y(y)$. This yields

$$\frac{1}{X(x)}\frac{d^2X}{dx^2}(x)+\frac{1}{Y(y)}\frac{d^2Y}{dy^2}=0$$

The usual Separation of Variables argument now tells us that

$$\frac{1}{X(x)}\frac{d^2X}{dx^2}(x)=C=-\frac{1}{Y(y)}\frac{d^2Y}{dy^2}$$

where C is a constant. This leads us to the following pair of ODEs

$$X''=CX \tag{3e}$$

$$Y''=-CY \tag{3f}$$

Next we plug (3d) into the boundary conditions (3c). This lead us to

$$\begin{aligned} 0&=X(0)\,Y(y) \quad , \quad \forall\, y\in[0,b] \quad \Longrightarrow \quad X(0)=0\\ 0&=X(a)\,Y(y) \quad , \quad \forall\, y\in[0,b] \quad \Longrightarrow \quad X(a)=0\end{aligned}$$

otherwise, we'd be forced to set $Y(y)=0$ for all $y\in[0,b]$ and the solution (3d) would be identically zero for all x and y. Now

$$X''=CX$$

has two different kinds of solutions, depending on whether C is postive or negative.

If C is positive, say $C=\lambda^2$, then the general soltuion of (3e) will be of the form

$$X(x)=c_1\cosh(\lambda x)+c_2\sinh(\lambda x)$$

To satisfy $X(0)=0$ we'd have to take $c_1=0$, but then we could not also satisfy $0=(a)=c_2\sinh(\lambda a)$ unless c_2 were also zero. We conclude that C can not be positive.

So we take $C=-\lambda^2<0$. Now

$$X''=-\lambda^2 C \qquad \Longrightarrow \qquad X=A\sin(\lambda x+\delta) \qquad \text{for some } A\in\mathbb{R}\text{, some } \delta\in[0,2\pi)$$

Imposing the first boundary condition

$$0=X(0)=A\sin(\delta) \quad \Longrightarrow \quad \delta=0$$

because setting $A=0$ would otherwise trivialize the solution. Setting $\delta=0$ and imposing the second boundary conditions leads to

$$0=X(a)=A\sin(\lambda a) \quad \Longrightarrow \quad \lambda a=n\pi \quad \Longrightarrow \quad \lambda=\frac{n\pi}{a} \quad , \quad n=1,2,3,\ldots$$

What we have so far is that

$$C=-\lambda^2=-\frac{n^2\pi^2}{a^2} \qquad , \qquad X(x)=A\sin\left(\frac{n\pi}{a}x\right)$$

We now impose the boundary condition $u(x,0)=0$ on our ansatz (3d):

$$0=u(x,0)=X(x)\,Y(0) \quad \Longrightarrow \quad Y(0)=0$$

But $Y(y)$ is also to satisfy (3f)

$$Y''=-CY=\frac{n^2\pi^2}{a^2}Y \quad \Longrightarrow \quad Y=c_1\cosh\left(\frac{n\pi}{a}y\right)+c_2\sinh\left(\frac{n\pi}{a}y\right)$$

The boundary condition $0 = Y(0)$ forces us to take $c_1 = 0$.

The preceding arguments have furnished us the following family of solutions of (3a), (3c) and the first condition of (3b):

$$\phi_n(x,y) = \sin\left(\frac{n\pi}{a}x\right)\sinh\left(\frac{n\pi}{a}y\right) \qquad , \qquad n = 1,2,3,\dots$$

Any linear combination of these functions will continuation to satisfy the PDE and the first three boundary conditions, and so we set

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{a}x\right)\sinh\left(\frac{n\pi}{a}y\right)$$

and try to impose the last boundary condition:

(3g)
$$g(x) = u(x,b) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{a}x\right)\sinh\left(\frac{n\pi}{a}b\right)$$

Multiplying both sides of (3g) by $\frac{a}{2}\sin\left(\frac{m\pi}{a}x\right)$ and then integrating over $[0,a]$ yields

$$\begin{aligned}
\frac{2}{a}\int_0^a \sin\left(\frac{m\pi}{a}x\right) g(x)\,dx &= \sum_{n=1}^{\infty}\frac{2}{a}\int_0^a c_n \sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{a}x\right)\sinh\left(\frac{n\pi}{a}b\right)dx \\
&= \sum_{n=1}^{\infty} c_n \sinh\left(\frac{n\pi}{a}b\right)\delta_{m,n} \\
&= c_n \sinh\left(\frac{m\pi}{b}b\right)
\end{aligned}$$

We conclude if the constants c_n are chosen so that

$$c_n = \frac{2}{a\sinh\left(\frac{n\pi}{a}b\right)}\int_0^a \sin\left(\frac{m\pi}{a}x\right) g(x)\,dx$$

then

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{a}x\right)\sinh\left(\frac{n\pi}{a}y\right)$$

will satisfy Laplace's equation and all four boundary conditions.

4.

(a) Apply Separation of Variables to reduce the problem of finding a solution the following PDE to that of solving a pair of ODEs.

(4a)
$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = 0$$

- Suppose

(4b)
$$\phi(r,\theta) = R(r)\,T(\theta)$$

Then substituting (4b) into (4a) yields

(4c)
$$T(\theta)\frac{d^2R}{dr^2}(r) + \frac{1}{r}T(\theta)\frac{dR}{dr}(r) + \frac{1}{r^2}R(r)\frac{d^2T}{d\theta^2}(\theta) = 0$$

Multiplying both sides of (4c) by $r^2/\left(T(\theta)\,R(r)\right)$ yields

$$\frac{r^2}{R(r)}\frac{d^2R}{dr^2}(r) + \frac{r}{R(r)}\frac{dR}{dr}(r) + \frac{1}{T(\theta)}\frac{d^2T}{d\theta^2}(\theta) = 0$$

or

(4d)
$$\frac{r^2}{R(r)}\frac{d^2R}{dr^2}(r) + \frac{r}{R(r)}\frac{dR}{dr}(r) = -\frac{1}{T(\theta)}\frac{d^2T}{d\theta^2}(\theta)$$

Since the right hand side of (4d) is independent of r so must be the left hand side; and since the left hand side of is independent of θ so must be the right hand side. So both sides are independent of r and θ; hence both sides equal a constant. Call this constant C. We then have

$$\frac{r^2}{R(r)}\frac{d^2R}{dr^2}(r) + \frac{r}{R(r)}\frac{dR}{dr}(r) = C = -\frac{1}{T(\theta)}\frac{d^2T}{d\theta^2}(\theta)$$

or

(4e)
$$\frac{r^2}{R(r)}\frac{d^2R}{dr^2}(r) + \frac{r}{R(r)}\frac{dR}{dr}(r) = C$$

(4f)
$$-\frac{1}{T(\theta)}\frac{d^2T}{d\theta^2}(\theta) = C$$

Multiplying (4e) by $R(r)$ and (4f) by $T(\theta)$ we obtain

(4g)
$$r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} - CR = 0 \quad ,$$

(4h)
$$\frac{d^2T}{d\theta^2} + CT = 0 \quad ;$$

a pair of ordinary differential equations for $R(r)$ and $T(\theta)$.

(b) Use the results of (a) to formulate an expression for the general solution of $\nabla^2\phi = 0$ on the disc $D = \left\{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 4\right\}$. (Hint: the general solution of $x^2y'' + xy' - \lambda^2 y = 0$ is $y(x) = c_1x^\lambda + c_2x^{-\lambda}$ if $\lambda \neq 0$, or $y(x) = c_1 + c_2\ln|x|$ if $\lambda = 0$.)

- Write $C = \lambda^2$.We have

$$\frac{d^2T}{d\theta^2} + \lambda^2 T = 0 \quad \Rightarrow \quad \Theta(\theta) = \begin{cases} a_\lambda\cos(\lambda\theta) + b_\lambda\sin(\lambda\theta) & \text{if } \lambda \neq 0 \\ a_0 + b_0\theta & \text{if } \lambda = 0 \end{cases}$$

$$r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} - \lambda^2 R = 0 \quad \Rightarrow \quad R(r) = \begin{cases} c_\lambda r^\lambda + d_\lambda r^{-\lambda} & \text{if } \lambda \neq 0 \\ c_0 + d_0\ln|r| & \text{if } \lambda = 0 \end{cases}$$

Products of these solutions (using the same values of λ) will be solutions of Laplace's equation. To get a more general solution we take general linear combinations of these types of solutions

$$u(r,\theta) = (a_0 + b_0\theta)(c_0 + d_0 \ln|r|) + \sum_{\lambda} (a_\lambda \cos(\lambda\theta) + b_\lambda \sin(\lambda\theta))\, c_\lambda r^\lambda + d_\lambda r^{-\lambda}$$

$$\sim A_0 + B_0\theta + C_0 \ln|r| + D_0\theta \ln|r| + \sum_{\lambda} \left(A_\lambda \cos(\lambda\theta)\, r^\lambda + B_\lambda \sin(\lambda\theta)\, r^\lambda + C_\lambda \cos(\lambda\theta)\, r^{-\lambda} + D_\lambda \sin(\lambda\theta)\, r^{-\lambda}\right)$$

(c) Can your result in part (b) be simplified by imposing regularity conditions on your solution? (Hint: yes. But what are the arguments?).

- If we demand the solutions are periodic with respect to θ (so that $u(r,\theta) = u(r,\theta+2\pi)$) we need to set $B_0 = D_0 = 0$ and restrict λ to be an integer.

$$u(r,\theta) = A_0 + C_0 \ln|r| + \sum_{n=1}^{\infty} \left(A_n \cos(n\theta)\, r^n + B_n \sin(n\theta)\, r^n + C_n \cos(n\theta)\, r^{-n} + D_n \sin(n\theta)\, r^{-n}\right)$$

  If we demand that the solutions remain continuous as $r \to 0$, we need to get rid of the solutions involving $\ln|r|$ and r^{-n}. Thus,

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} (A_n \cos(n\theta)\, r^n + B_n \sin(n\theta)\, r^n)$$

  would be our simplified general solution.