Math 4233
SOLUTIONS TO FIRST EXAM
June 28, 2012

1.

(a) Find the eigenvectors and eigenvalues of the following matrix

$$\left[\begin{array}{cc} 2 & -1 \\ 1 & 2 \end{array}\right]$$

- To find the eigenvalues, we determine the roots of the characteristic polynomial of the matrix:

$$\begin{aligned} 0 &= p(\lambda) := \det\left(\left[\begin{array}{cc} 2-\lambda & -1 \\ 1 & 2-\lambda \end{array}\right]\right) = \left(2-\lambda^2\right) - 1 = \lambda^2 - 4\lambda + 3 \\ \Rightarrow & \quad \lambda = \frac{4 \pm \sqrt{(-4)^2 - 4(1)(3)}}{2} = \frac{4 \pm \sqrt{-4}}{2} = 2 \pm i \end{aligned}$$

So we have two complex eigenvalues: $\lambda = 2+i$, $2-i$.
Next, we find the corresponding eigenvectors:

$$\begin{aligned} \text{eigenvectors for } \lambda &= 2+i \iff NullSp\left(\left[\begin{array}{cc} 2-(2+i) & -1 \\ 1 & 2-(2+i) \end{array}\right]\right) \\ &= NullSp\left(\left[\begin{array}{cc} -i & -1 \\ 1 & -i \end{array}\right]\right) = NullSp\left(\left[\begin{array}{cc} 1 & -i \\ 0 & 0 \end{array}\right]\right) = span\left(\left[\begin{array}{c} 1 \\ -i \end{array}\right]\right) \end{aligned}$$

and

$$\begin{aligned} \text{eigenvectors for } \lambda &= 2-i \iff NullSp\left(\left[\begin{array}{cc} 2-(2-i) & -1 \\ 1 & 2-(2-i) \end{array}\right]\right) \\ &= NullSp\left(\left[\begin{array}{cc} i & -1 \\ 1 & i \end{array}\right]\right) = NullSp\left(\left[\begin{array}{cc} 1 & i \\ 0 & 0 \end{array}\right]\right) = span\left(\left[\begin{array}{c} 1 \\ i \end{array}\right]\right) \end{aligned}$$

So

$$\begin{aligned} \lambda_1 &= 2+i \quad , \quad \mathbf{v}_{\lambda_1} = \left[\begin{array}{c} 1 \\ -i \end{array}\right] \\ \lambda_2 &= 2-i \quad , \quad \mathbf{v}_{\lambda_2} = \left[\begin{array}{c} 1 \\ i \end{array}\right] \end{aligned}$$

(b) Find the eigenvectors and generalized eigenvectors of

$$\left[\begin{array}{cc} 1 & 2 \\ 0 & 1 \end{array}\right]$$

- First we find the eigenvalue(s) and eigenvector(s).

$$0 = \det\left(\left[\begin{array}{cc} 1-\lambda & 2 \\ 0 & 1-\lambda \end{array}\right]\right) = (1-\lambda)^2 \quad \Rightarrow \quad \lambda = 1 \text{ (only one eigenvalue)}$$

$$\text{eigenvectors for } \lambda = 1 = NullSp\left(\left[\begin{array}{cc} 1-1 & 2 \\ 0 & 1-1 \end{array}\right]\right) = NullSp\left(\left[\begin{array}{cc} 0 & 2 \\ 0 & 0 \end{array}\right]\right) = NullSp\left(\left[\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array}\right]\right) = span\left(\left[\begin{array}{c} 1 \\ 0 \end{array}\right]\right)$$

So we have one eigenvalue $\lambda = 1$ with one (linearly independent) eigenvector $\mathbf{v}_{\lambda=1} = \left[\begin{array}{c} 1 \\ 0 \end{array}\right]$. Since the original matrix is 2x2, there must be an additional *genralized eigenvector* $\boldsymbol{\eta}$. It will be the solution of

$$\begin{bmatrix} 1-\lambda & 2 \\ 0 & 1-\lambda \end{bmatrix} \boldsymbol{\eta} \quad = \mathbf{v} \quad {}_\lambda \quad \text{or} \quad \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} 2\eta_2 = 1 \\ 0 = 0 \end{cases}$$

$$\Rightarrow \qquad \boldsymbol{\eta} = \begin{bmatrix} 0 \\ \frac{1/}{2} \end{bmatrix} + \begin{bmatrix} \eta_1 \\ 0 \end{bmatrix}$$

with η_1 arbitary.

2. Find the general solution of the following homogeneous linear system

$$\begin{aligned}\frac{dx_1}{dt} &= x_1 + 2x_2 \\ \frac{dx_2}{dt} &= 2x_1 - 2x_2\end{aligned}$$

- The coefficient matrix for this homogeneous linear system is

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$$

Its eigenvalues and eigenvectors are

$$\begin{aligned}\lambda_1 &= 2 \quad , \quad \mathbf{v}_{\lambda_1} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \\ \lambda_2 &= -3 \quad , \quad \mathbf{v}_{\lambda_2} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}\end{aligned}$$

We thus have the following two fundamental solutions $\mathbf{x}_1(t) = e^{2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad , \quad \mathbf{x}_2(t) = e^{-3t} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

One can express the general solution either as a general linear combination of the two fundamental solutions

$$\mathbf{x}(t) = c_1 e^{2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 2c_1 e^{2t} + c_2 e^{-3t} \\ c_1 e^{2t} - 2c_2 e^{-3t} \end{bmatrix}$$

or in terms of the fundamental matrix

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\,\mathbf{c} = \begin{bmatrix} 2e^{2t} & e^{-3t} \\ e^{2t} & -2e^{-3t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

3. Solve the following nonhomogenous linear system

$$\begin{aligned}\frac{dx_1}{dt} &= -2x_1 + 3x_2 + e^t \\ \frac{dx_2}{dt} &= -x_1 + 2x_2 + t\end{aligned}$$

given that the general solution to the corresponding homogeneous problem is

$$\mathbf{x}_o(t) = c_1 e^{-t}\left[\begin{array}{c} 3 \\ 1 \end{array}\right] + c_2 e^t \left[\begin{array}{c} 1 \\ 1 \end{array}\right]$$

- The Fundamental Matrix for the homogeneous problem can be read off its general solution it is

$$\mathbf{\Phi}(t) = \left[\begin{array}{cc} 3e^{-t} & e^t \\ e^{-t} & e^t \end{array}\right]$$

  and its inverse is

$$\mathbf{\Phi}^{-1}(t) = \frac{1}{3-1}\left[\begin{array}{cc} e^t & -e^t \\ -e^{-t} & 3e^{-t} \end{array}\right]$$

  Also, the *driving term* on the right hand side of the inhomogenous system is

$$\mathbf{g}(t) = \left[\begin{array}{c} e^t \\ t \end{array}\right]$$

  We can compute the general solution to the inhomogeneous system via the formula

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\int_0^t \mathbf{\Phi}^{-1}(s)\,\mathbf{g}(s)\,ds + \mathbf{\Phi}(t)\,\mathbf{c}$$

  Let's first compute

$$\int_0^t \mathbf{\Phi}^{-1}(s)\,\mathbf{g}(s)\,ds = \frac{1}{2}\int_0^t \left[\begin{array}{cc} e^s & -e^s \\ -e^{-s} & 3e^{-s} \end{array}\right]\left[\begin{array}{c} e^s \\ s \end{array}\right] ds = \frac{1}{2}\int_0^t \left[\begin{array}{c} e^{2s} - se^s \\ -1 + 3se^s \end{array}\right] dx$$

  Using

$$\int_0^t e^{as} ds = \frac{1}{a}e^{at} - \frac{1}{a} \quad , \quad \int_0^t se^{as} ds = \left(\frac{s}{a}e^{as}\right)\Big|_0^t - \left(\frac{1}{a^2}e^{as}\right)\Big|_0^t = te^{at} - \frac{1}{a^2}e^{\alpha t} + \frac{1}{a^2}$$

  we get

$$\begin{aligned}\frac{1}{2}\int_0^t \left(e^{2s} - se^s\right) ds &= \frac{1}{2}e^t - \frac{1}{2}te^t + \frac{1}{4}e^{2t} - \frac{3}{4} \\ \frac{1}{2}\int_0^t \left(-1 + 3se^s\right) ds &= \frac{3}{2}te^t - \frac{3}{2}e^t - \frac{1}{2}t + \frac{3}{2}\end{aligned}$$

4. Consider the autonomous nonlinear system : $\dfrac{dx}{dt} = x(2-x-y) \quad , \quad \dfrac{dy}{dt} = (1-y)(2+x)$

(a) Determine the critical points of this system.

- We are looking for points (x, y) where
$$\begin{aligned} 0 &= F_1(x,y) = x(2-x-y) \\ 0 &= F_2(x,y) = (1-y)(2+x) \end{aligned}$$
The first equation requires either $x = 0$ or $y = -x+2$. Substituting $x = 0$ into the second equation yields $0 = (1-y)(2) \quad \Rightarrow \quad y = 1$. On the other hand, substituting $y = -x+2$ into the second equation yields $0 = (1+x-2)(2+x) \quad \Rightarrow \quad x = -2, 1$. Thus we have have a total of three solutions
$$(0,1) \quad , \quad (-2,4) \quad , \quad (1,1)$$

(b) Determine the corresponding linear systems near these critical points and discuss the stability of solutions near these critical points.

- We have
$$\mathbf{DF} = \begin{bmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{bmatrix} = \begin{bmatrix} 2-2x-y & -x \\ 1-y & -x-2 \end{bmatrix}$$
At the critical point $(0,1)$, $\mathbf{DF}$ becomes
$$\mathbf{DF} = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$$
Clearly, the eigenvalues of this matrix are 1 and -2. Because the eigenvalue 1 is positive, $(0,1)$ will be an unstable critical point.
- At the critical point $(-2,4)$, $\mathbf{DF}$ becomes
$$\mathbf{DF} = \begin{bmatrix} 2 & 2 \\ -3 & 0 \end{bmatrix}$$
, eigenvalues: $i\sqrt{5}+1, 1-i\sqrt{5}$Now
$$\det(\mathbf{DF} - \lambda\mathbf{I}) = (2-\lambda)(0-\lambda) + 6 = \lambda^2 - 2\lambda + 6$$
So the eigenvalues of $\mathbf{A}$ are
$$\lambda = \frac{2 \pm \sqrt{4-24}}{2} = 1 \pm \frac{\sqrt{-20}}{2} = 1 \pm \sqrt{5}i$$
Because both eigenvalues have a positive real part, this critical point will be unstable.
- At the critical point $(1,1)$, we have
$$\mathbf{DF} = \begin{bmatrix} -1 & -1 \\ 0 & -3 \end{bmatrix}$$
The eigenvalues of this matrix are the roots of
$$0 = \det\left(\begin{bmatrix} -1-\lambda & -1 \\ 0 & -3-\lambda \end{bmatrix}\right) = (\lambda+1)(\lambda+3) \quad \Rightarrow \quad \lambda = -1, -3$$
Since both eigenvalues are negative, $(1,1)$ will be an asymptotically stable critical point.