LECTURE 19

Regular Singular Points and Generalized Power Series

In the preceding lecture, we reviewed the construction of power series solutions of second order, linear, homogeneous ODE

$$y'' + p\,(x)\,y' + q\,(x)\,y$$

that worked in great generality; so long as we avoided the singular points of $p\,(x)$ and $q\,(x)$.

We'll now turn our attention to finding solution that are valid right up to a singular point of the differential equation. However, here again we are going to have some limitations. The main one being that if a singularity is too drastic, we may still be unable to construct solutions.

So let me dilineate right away the situation in which our generalized power technique will succeed.

1. Regular Singular Points

Definition 19.1. *A differential equation of the form*

$$y'' + p(x)y' + q(x)y = 0 \tag{1}$$

is said to have a ***singular point*** *at x_o if either*

$$\lim_{x\to x_o} p(x) \tag{2}$$

or

$$\lim_{x\to x_o} q(x) \tag{3}$$

does not exist.

Definition 19.2. *Suppose x_o is a singular point of*

$$y'' + p(x)y' + q(x)y = 0 \quad .$$

x_o is said to be a ***regular singular point*** *(of this differential equation) if the singularity of $p(x)$ is no worse than*

$$\frac{1}{x - x_o}$$

and the singularity of $q(x)$ is no worse than

$$\frac{1}{(x - x_0)^2} \quad .$$

More precisely, x_o is a regular singular point if both

$$\lim_{x\to x_o} (x - x_o)\,p(x) \tag{4}$$

and

$$\lim_{x\to x_o} (x - x_o)^2\,q(x) \tag{5}$$

exist. Otherwise, x_o is called an ***irregular singular point****.*

EXAMPLE 19.3. The differential equation

$$y'' + \frac{3}{(x-1)(x+1)^2} y' + \frac{2x+1}{(x-2)^2(x+2)(x-1)^3} y = 0$$

has singular points at $x = 1, -1, 2, -2$.

Now

Singular Point x_o	$\lim_{x \to x_o} (x - x_o) p(x)$	$\lim_{x \to x_o} (x - x_o)^2 q(x)$	Type
1	$\frac{3}{4}$	∞	irregular
-1	∞	0	irregular
2	0	$\frac{5}{4}$	regular
-2	0	0	regular

So $x = \pm 1$ are irregular singular points and $x = \pm 2$ are regular singular points.

EXAMPLE 19.4. Identify and classify the singular points of

$$x^2(1-x^2)^2 y'' + x(1+x)^2 y' + (1-x) y' \quad .$$

In this case, when we divide by $x^2(1-x^2)^2$ to put the equation in standard form, we have

$$p(x) = \frac{x(1+x)(1+x)}{x^2(1+x)^2(1-x)^2} = \frac{1}{x(1-x)^2}$$

and

$$q(x) = \frac{(1-x)}{x^2(1+x)^2(1-x)^2} = \frac{1}{x^2(1+x)^2(1-x)} \quad .$$

Thus, we have regular singular points at $x = 0, -1$ and an irregular singular point at $x = 1$.

The point of this notion of *regular singular point* is that the generalized power series technique that we develop in the next section is only works for finding solution near regular singular points.

2. The Generalized Power Series Method

2.1. Generalized Power Series.

DEFINITION 19.5. *A* ***generalized power series*** *about* x_0 *is a formal series of the form*

$$(x - x_0)^r \sum_{n=0}^{\infty} a_n (x - x_0) = \sum_{n=0}^{\infty} a_n (x - x_0)^{n+r}$$

What makes such a generalized power series distinct from an ordinary power series is that we allow the parameter r to be any real number.

So if, for example, $r = -2$, the corresponding series would represent an infinite sum that started off as

$$a_0 (x - x_0)^{-2} + a_1 (x - x_0)^{-1} + a_2 + a_3 (x - x_0) + a_4 (x - x_0)^2 + \cdots$$

As x approaches the "expansion point" x_0 the leading term dominates the generalized power series behaves like

$$\sum_{n=0}^{\infty} a_n (x - x_0)^{n+r} \approx \frac{a_0}{(x - x_0)^2} \qquad \text{for } |x - x_0| << 1$$

In this example, while the generalized power series about x_0 is undefined at x_0, it could nevertheless converge to a legitimate function that is valid for all x except x_0. In this way we have a template for solutions of a differential equation that might be singular at x_0, and yet gives us a way of understanding solutions right up to x_0.

CONVENTION 19.6. *If*

$$\sum_{n=0}^{\infty} a_n (x - x_0)^{n+r} \tag{6}$$

is a generalized power series, we shall always assume that the initial coefficient a_0 is non-zero. This just means we insist on interpreting

$$a_0 (x - x_0)^r$$

as the leading non-zero term of (6)

This convention will allow us to talk directly about the leading term of a generalized power series without being forced to say things like, "$(x - x_0)^r$ is the leading power of $(x - x_0)$ unless $a_0 = 0$, in which case $(x - x_0)^{r+1}$ is the leading power of $(x - x_0)$; unless, both $a_0 = a_1 = 0$ in which case $(x - x_0)^{r+2}$ is the leading power of $(x - x_0)$; unless ...."

2.2. The generalized power series method. The basic technique of the generalized power series method is pretty much the same the usual power series method.

(i) one poses a generalized power series as an ansatz for a solution,
(ii) one next plugs that ansatz into the differential equation,
(iii) one performs a series of power series manipulations in order to express the differential equation as a condition saying that particular generalized power series must equal zero,
(iv) from that condition, one deduces the restrictions on and recursion relations amongst the coefficients a_n.
(v) Finally, one systematically constructs the solution from the first coefficients

However, there are a several notable differences between the two techniques.

First of all, it will turn out that condition uncovered in step (iii) will also put a restriction on the parameter r.

Secondly, instead of being able to solve for all the higher coefficients in terms of a_0 and a_1 (effectively producing two linearly independent solutions of the differential equation), the restrictions on recursion relations amongst the coefficients a_n as exhibited in step (iv), will allow all coefficients to be computed from just a_0. Thus, we'll only be guaranteed one linearly independent solution by this technique.

Let me illustrate both the generalized power series method and its deviations from the ordinary power series method with a simple example.

EXAMPLE 19.7.

$$xy'' - y = 0$$

- This differential equation has a regular singular point at $x = 0$ (If you put it in standard form, you get $p(x) = 0$ and $q(x) = \frac{1}{x}$). So we'll try constructing a generalized power series solution about $x = 0$. Thus, we first set

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$$

We next calculate the generalized power series expression for the term xy'' :

$$\begin{aligned} xy'' &= x\sum_{n=0}^{\infty}(n+r)(n+r-1)\,a_n x^{n+r-2} \\ &= \sum_{n=0}^{\infty}(n+r)(n+r-1)\,a_n x^{n+r-1} \\ &= \sum_{n=-1}^{\infty}(n+r+1)(n+r)\,a_{n+1}x^{n+r} \\ &= r(r-11)\,a_0x^{r-1}+\sum_{n=0}^{\infty}(n+r+1)(n+r)\,a_{n+1}x^{n+r} \end{aligned}$$

And now we can express the differential equation as

$$\begin{aligned} 0 &= xy''-y \\ &= r(r+1)\,a_0x^{r-1}+\sum_{n=0}^{\infty}(n+r+1)(n+r)\,a_{n+1}x^{n+r}-\sum_{n=0}^{\infty}a_nx^{n+r} \\ &= r(r-a)\,a_0x^{r-1}+\sum_{n=0}^{\infty}\left[(n+r+1)(n+r)\,a_{n+1}-a_n\right]x^{n+r} \end{aligned}$$

Setting the total coefficient of each distinct power of x equal to zero we get

$$r(r-1)\,a_0=0 \tag{*}$$

and

$$(n+r+1)(n+r)\,a_{n+1}-a_n=0 \qquad , \qquad n=0,1,2,3 \tag{**}$$

Because, by convention, we are assuming that $a_0 \neq 0$, we can deduce from the first equation (*) that the parameter r (which under our convention corresponds to the leading power of x in the generalized power series solution) must satisfy

$$r(r-1)=0 \quad\Rightarrow\quad r=0,1$$

Something similar will happen in other examples; *requiring that the total coefficient of the lowest power of* $(x-x_0)$ *that occurs in the generalized power series expression of the differential equation will give us a condition on the paramenter* r. This condition is called the *indicial equation* for the differential equation (at the singular point).

The second equation provides us with a recursion relation

$$a_{n+1}=\frac{a_n}{(n+r+1)(n+r)}$$

which will allow us to compute each of the coefficients $a_1,a_2,a_3,\ldots$ successively from the initial coefficient a_0.

In fact, we'll get two sets of recursion relations, because we have two choices for r.

$r=1$ The recursion relations in this case becomes

$$a_{n+1}=\frac{a_n}{(n+2)(n+1)}$$

and so

$$\begin{aligned} a_1 &= \frac{a_0}{2} \\ a_2 &= \frac{a_1}{6}=\frac{a_0}{12} \\ a_3 &= \frac{a_2}{12}=\frac{a_0}{144} \end{aligned}$$

Thus, the solution corresponding to $r = 1$ is

$$\begin{aligned} y(x) &= \sum_{n=0}^{\infty} a_n x^{n+1} \\ &= a_0 x + \frac{a_0}{2} x^2 + \frac{a_0}{12} x^3 + \frac{a_0}{144} x^4 + \cdots \\ &= a_0 \left(x + \frac{1}{2} x^2 + \frac{1}{12} x^3 + \frac{1}{144} x^4 + \cdots \right) \end{aligned}$$

$r = 0$ The recursion relation in this case becomes

$$a_{n+1} = \frac{a_n}{n(n+1)}$$

However, the first time we try to apply it

$$a_1 = \frac{a_0}{(0)(0+1)}$$

we get nonsense, because we can't divide by zero. This phenomenon I briefly alluded to above when I said we are only guaranteed one solution by the generalized power series method.

In the example at hand, we get only one valid solution via our generalized power technique, it is the solution corresponding to the larger root of the indicial equation.

Here is what happens in general. Suppose

$$0 = \sum_{n=n_0}^{\infty} F_n(n, r, a_*) x^n$$

is the form of the differential equation once it has been expressed as a single generalized power series. The total coefficient F_{n_0} of the lowest power of x will be of the form

$$(r(r-1) + \beta r + \gamma) a_0$$

and since $a_0 \neq 0$, it will furnish a quadratic equation for r; this is the *indicial equation.*[1]

In general (since it is always a quadratic equation), the indicial equation will have two roots r_1 and r_2. The situation we encountered in the example above where we could construct one solution as a generalized power series occurs precisely when r_1 and r_2 are both real and differ by an integer.

- If $r_1, r_2 \in \mathbb{R}$ and $r_1 - r_2 \in \mathbb{N}$ then we get only one solution $y_1(x)$ via the generalized power series technique. It will be the solution corresponding the using the recursion relations for the larger of the two roots.

(In the example above, we constructed a solution for the root $r = 1$, but we were unable to construct a solution for the root $r = 0$).

[1] In fact, the general form of a 2^{nd} order linear differential with a regular singularity at $x = 0$ will be

$$y'' + \frac{q(x)}{x} y' + \frac{p(x)}{x^2} y = 0$$

or

$$x^2 y'' + x p(x) y' + q(x) y = 0$$

If one replaces $p(x)$ and $q(x)$ by their power (Taylor) series

$$p(x) = \sum_{n=0}^{\infty} p_n x^n \quad , \quad q(x) = \sum_{n=0}^{\infty} q_n x^n$$

and applies the generalized power series technique, one obtains as the corresponding indicial equation

$$r(r-1) + p_0 r + q_0 = 0$$

On the other hand, if r_1 and r_2 are not real numbers, or if their difference is not an integer, then we will in fact get an independent power series solution for each root; one using the recursion relations $r = r_1$, the other using the recursion relations with $r = r_2$

3. The Generalized Power Series Method in Detail

Here I'll try to describe the generalized power series method in complete detail, focusing especially on how to one arrives at the two independent solutions needed for a general solution. To keep our notation as simple as possible, I'll assume the singular point is at $x = 0$. It should be clear that the substitutions $x \to (x - x_0)$ in the expressions below would suffice to handle the case where we had a singular point at $x = x_0$.

Suppose

$$x^2 y'' + xp(x) y' + q(x) \tag{7}$$

has a regular singular point at $x = 0$ (which will happen if $p(0) \neq 0$ or $q(0) \neq 0$ or $q'(0) \neq 0$). Write

$$p(x) = \sum_{m=0} p_m x^m \qquad , \qquad q(x) = \sum_{m=0}^{\infty} q_m x^m \tag{8}$$

We now pose

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \qquad , \qquad a_0 \neq 0 \tag{9}$$

as an ansatz for the solution of (7). Inserting these series expressions for y, q and p into (8) yields

$$0 = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r} + \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (n+r) a_n p_m x^{m+n+r} + \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_n q_m x^{n+m+r} = 0 \tag{(10)}$$

Observe that the lowest power of x that occurs each in the three series terms of (10) is x^r (corresponding to the $m = 0$ and $n = 0$ terms). Its total coefficient is

$$r(r-1) a_0 + r p_0 a_0 + q_0 a_0$$

This coefficient must equal zero if we are to maintain the condition (10) for all x. Since by convention $a_0 \neq 0$, we are led to the following condition

$$r(r-1) + r p_0 + q_0 = 0 \tag{11}$$

This is the *indicial equation* for (7). The roots of the indicial requation are the only values for r for which the generalized power series can provide a (formal) solution to (7).

Then the generalized power series expression of the (7) can be succinctly written in terms of F and the power series expressions for the coefficient functions $p(x)$ and $q(x)$.

First of all, (10) can be rewritten as

$$0 = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r} + \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \left[(k+r) p_m + q_m \right] a_k x^{m+k+r} \tag{12}$$

Now let

$$\begin{aligned} n &= k + m \\ m &= n - k \end{aligned}$$

and collect the total coefficient of x^{n+r} in the double sum

$$\sum_{n'=0}^{\infty} \sum_{m=0}^{\infty} \left[(n'+r) p_{n'} + q_{n'} \right] a_k x^{m+n'+r} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \left[(k+r) p_{n-k} + q_{n-k} \right] a_k x^{n+r}$$

This manipulation allows us to rewrite (12) as

$$0=\sum_{n=0}^{\infty}\left[(n+r)(n+r-1)a_n+\sum_{k=0}^{n}\left[(k+r)p_{n-k}+q_{n-k}\right]a_k\right]x^{n+r}$$

Let us now peel off the total coefficient of x^r (the leading term) and separate the term involving the highest coefficient a_n in the total coefficient of x^{n+r} :

$$\begin{aligned}0&=\left[r(r-1)+p_0r+q_0\right]a_0x^r\\&\quad+\sum_{n=1}^{\infty}\left[(n+r)(n+r-1)a_n+((n+r)p_0+q_0)a_n\right.\\&\qquad\left.+\sum_{k=0}^{n-1}\left[(k+r)p_{n-k}+q_{n-k}\right]a_k\right]x^{n+r}\end{aligned}$$

or

$$0=a_0F(r)x^r+\sum_{n=1}^{\infty}\left[F(r+n)a_n+\sum_{k=0}^{n-1}a_k\left((r+k)p_{n-k}+q_{n-k}\right)\right]x^{r+n}\tag{13}$$

where

$$F(r)\equiv r(r-1)+rp_0+q_0$$

is the polynomial on the left hand side of the indicial equation.

From (13) we can read off both the indicial equation

$$0=F(r)=r(r-1)+p_0r+q_0\tag{14}$$

and the recursion relations

$$a_n=\frac{1}{F(r+n)}\sum_{k=0}^{n-1}a_k\left((r+k)p_{n-k}+q_{n-k}\right)\tag{15}$$

Note that the indicial equation (14) is a quadratic equation; as such it has either one real root, two real roots, or a pair of complex conjugate roots. Via the Fundamental Theorem of Algebra, we can write[2]

$$F(r)=(r-r_1)(r-r_2)$$

For the purpose of delineating the variations of generalized power series method, it is more useful to regard the roots r_1,r_2 of the indicial equations as being one of the following three types.

Case 1: There are two distinct roots r_1,r_2 of (11) and r_1-r_2 is not an integer
Case 2: There is only one real root r of (11).
Case 3: There are two real roots r_1,r_2 of (11) and $r_1-r_2=N\in\{1,2,3,\dots\}$

You see, as in our initial example, the one thing that might prevent us from solving the recursion relations (15) successively for $a_1,a_2,a_3,\dots$ is the circumstance where divisors $F(r+1)$, $F(r+2)$, $F(r+3),\dots$ on the right hand side are never equal to zero.

[2]Via the quadratic formula

$$\begin{aligned}r_1&=\frac{1-p_0+\sqrt{(1-p_0)^2-4q_0}}{2}\\r_2&=\frac{1-p_0+\sqrt{(1-p_0)^2-4q_0}}{2}\end{aligned}$$

3.1. Case 1: $r_1 - r_2 \notin \mathbb{Z}$**.** Once we choose $r = r_1$ or r_2, we are assured

$$F(r_i + k) \neq 0 \qquad \text{for } k = 1, 2, 3, \ldots$$

For otherwise we would have two roots of $F(r)$ that differ by an integer k, and that situation is exactly what we've ruled in Case (1). Thus, in Case (1), the recursion relations (15) will be solvable for both $r = r_1$ and $r = r_2$.

Thus, so long as $r_1 - r_2 \notin \mathbb{Z}$, one gets two distinct sets of recurions relations from (13) by specializing to $r = r_1$ and $r = r_2$. Both sets of recurion relations can be solved iteratively so that coefficients $a_1(r_i), a_2(r_i), \ldots$ can be expressed as some constant times a_0. In this way, we'll arrive at two linearly independent solutions

$$y_1(x) = \sum_{n=0}^{\infty} a_n(r_1) x^{n+r_1}$$
$$y_2(x) = \sum_{n=0}^{\infty} a_n(r_2) x^{n+r_2}$$

3.2. Case 2: $r_1 = r_2 \in \mathbb{Z}$**.** In this case, $F(r)$ has only one distinct root, call it r_1. But then

$$F(r_1) = 0 \quad \Rightarrow \quad F(r_1 + k) \neq 0 \qquad \text{for } k = 1, 2, 3, \ldots$$

and the recursion relations are solvable, and a generalized power series solution can be found. However, this time we having only a single root of the indicial equation, we get only one independent generalized power series solution. From the general theory of second order linear differential equations, we must expect that the general solution will require a second independent solution.

To find a second independent solution, there's several things one could do. First of all, there is a fomular *Reduction of Order*, that, given one solution $y_1(x)$ of a homogeneous second order linear ODE of the form (7) allows one to compute a second linearly independent solution:

$$y_2(x) = y_1(x) \int \frac{1}{y_1(s)^2} \exp\left[-\int^s t\, p(t)\, dt\right] ds \tag{16}$$

This might not look practical given that the solution $y_1(x)$ found as above will be a generalized power series rather than an explicit function of x; However, there are nice closed formulas for expressing products and powers of (formal) generalized power series as other generalized power series. One could thus re-express $[y_1(x)]^{-2}$ as a generalized power series and then compute the integral formally, term-by-term.

A second approach is as follows. Since we have only one root r_1 of the $F(r)$ and because $F(r)$ is monic, we must in fact have

$$F(r) = (r - r_1)^2 \tag{17}$$

and equation (13), which remember is the generalized power series expression of the original differential equation,

(18)

$$\left(x^2 \frac{d}{dx^2} + x p(x) \frac{d}{dx} + q(x)\right) \sum_{n=0}^{\infty} a_n x^{n+r} = a_0 (r - r_1)^2 x^r + \sum_{n=1}^{\infty} \left[(r + n - r_1)^2 a_n + \sum_{k=0}^{n-1} a_k \left((r+k) p_{n-k} + q_{n-k}\right)\right] x^{r+n}$$

Suppose that instead of insisting that r be the root of the indicial equation, we instead regard r as a free parameter, and then use the vanishing of the total coefficients of x^{n+r} to relate the higher coefficients $a_1, a_2, \ldots,$ to a_0 and r. That is to say, let's say we have found functions $a_n(r)$ by iteratively computing

$$a_n(r) = -\frac{1}{(r + n - r^1)^2} \sum_{k=0}^{n-1} a_k(r) \left((r+k) p_{n-k} + q_{n-k}\right) \qquad , \qquad n = 1, 2, 3, \ldots \tag{19}$$

Then, for these choices of $a_n(r)$, (18) becomes

$$= a_0 (r - r_1)^2 x^r + 0$$

and, of course, if we *now* set $r = r_1$, we get $0 = 0$, meaning the differential equation is solved. But what we did here, is solve the recursion relations first for general r, and then in the end require that r be the root of the indicial equation as a different route to the same solution. In other words, if we determine the $a_n(r)$ via (19) and write

$$\phi(x, r) = \sum_{n=0}^{\infty} a_n(r)\, x^{n+r} \tag{20}$$

then, $\phi(x, r)$ wil be a solution of (7) precisely when $r = r_1$; but more generally, $\phi(x, r)$ will satisfy

$$\left(x^2 \frac{d}{dx^2} + xp(x)\frac{d}{dx} + q(x)\right)\phi(x, r) = a_0 (r - r_1)^2 x^r + 0 \tag{21}$$

What does this tell us? Well, if we differentiate both sides of (21) with respect to r and then evaluate the result at $r = r_1$, we get

$$\left(x^2 \frac{d}{dx^2} + xp(x)\frac{d}{dx} + q(x)\right)\frac{\partial\phi}{\partial r}(x, r_1) = a_0 2 (r - r_1) x^r\big|_{r=r_1} = 0$$

In other words, both $\phi(x, r_1)$ and $\frac{\partial\phi}{\partial r}(x, r_1)$ will be solutions of

$$x^2 \frac{dy}{dx^2} + xp(x)\frac{dy}{dx} + q(x)\, y = 0$$

Now, given (20),

$$\frac{\partial\phi}{\partial r}(x, r_1) = \frac{\partial}{\partial r} x^r \left(a_0 + \sum_{n=1}^{\infty} a_n(r)\, x^n\right)\Bigg|_{r=r_1} = (x^{r_1}\ln|x|)\sum_{n=0}^{\infty} a_n(r)\, x^n + x^{r_1}\sum_{n=1}^{\infty} a_n'(r_1)\, x^n$$

$$= \phi_1(x, r_1)\ln|x| + x^{r_1}\sum_{n=1}^{\infty} a_n'(r_1)\, x^n$$

and so we can use

$$y_1(x) = \sum_{n=0}^{\infty} a_n(r_1)\, x^{n+r_1}$$

$$y_2(x) = y_1 \ln[x] + \sum_{n=1} a_n'(r_1)\, x^{n+r_1}$$

as two linearly independent solutions of the original ODE (7).

Notice that in this case, it is clear the second solution $y_2(x)$ has a logarithmic singularity at $x = 0$. This is often good to know, especially if we're looking only for solutions that are well-behaved at $x = 0$.

3.3. Case 3: $r_1 - r_2 \in \mathbb{Z}$**.** Let r_1 be larger for the two roots so that $r_1 - r_2 = N \in \mathbb{Z}_{>0}$. In this situation the larger root of the indicial equation will alway lead to a set of recursion relations that can be solved to express each coefficient $a_1(r_1), a_2(r_1), \ldots$ in terms of a_0; this is because

$$F(r_1 + k) \neq 0 \qquad , \quad k = 1, 2, 3, \ldots$$

since the only other root of $F(r) = 0$ is $r_2 = r_1 - N$. Thus, setting $r = r_1$ and solving the recursion relations (15) we'll be able to construct one solution

$$y_1(x) = \sum_{n=0}^{\infty} a_n(r_1)\, x^{n+r_1}$$

of the differential equation. However, when one tries doing the same thing for the smaller root r_2, well then

$$F(r_2 + N) = F(r_1) = 0$$

and we won't be able to solve for $a_N(r_2)$.

Nevertheless, it is still possible to find a second linearly independent solution. In this case, one makes the following ansatz for a second solution

$$y_2(x) = ay_1(x)\ln|x| + \left(1 + \sum_{k=0}^{\infty} c_k x^{k+r_3}\right)$$

and then plugs into the differential equation to determine an appropriate choice for the constants a and $b_1, b_2, \ldots$.

Alternatively, the coefficients c_k can be determined from the r-dependent coefficients $a_n(r)$ given by formula (19) via

$$c_k = \frac{d}{dr}\left((r - r_2)\, a_n(r)\right)\bigg|_{r=r_2}$$

and the constant a is given by

$$a = \lim_{r \to r_2}\left((r - r_2)\, a_n(r)\right)$$

It may happen that the constant a multiplying the logarithmic term is 0. In this case, the second solution needn't have a singularity at $x = 0$.