LECTURE 17

Sturm-Liouville Theory and Special Functions

1. Examples of Sturm-Liouville Functions

Recall that a Sturm-Liouville is an ODE/BVP of the form

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right]+q(x)y+\lambda r(x)y=0 \qquad , \qquad a\leq x\leq b$$
$$a_1y(a)+a_1y'(a)=0$$
$$b_1y(b)+b_2y'(b)=0$$

Here we will relax on assumptions on coefficient functions $p(x)$ and $q(x)$, requirely only that they are either strictly non-negative or strictly non-positive on the open interval (a,b).

1.1. Sine/Cosine Functions.

Taking

$$p(x)=1=r(x) \quad , \quad q(x)=0$$

the Sturm-Liouville ODE becomes

$$y''+\lambda y=0$$

If we take

$$a=0 \quad , \quad b=1$$

and

$$a_1=b_1=1 \qquad , \qquad a_2=b_2=0$$

then we find we have no solutions unless $\sqrt{\lambda}=n\pi$, $n=0,1,2,\ldots$ and in that case

$$\lambda=n^2\pi^2 \qquad \Longrightarrow \qquad \phi_n=\sin(n\pi x)$$

are the Sturm-Liouville eigenfuctions. The orthogonality property of the Sturm-Liouville eigenfunctions leads to

$$2\int_0^1\sin(n\pi x)\sin(m\pi x)\,dx=\delta_{m,n}$$

Taking $a=0$, $b=0$ and

$$a_1=b_1=0 \qquad , \qquad a_2=b_2=1$$

we again find no solutions unless $\sqrt{\lambda}=n\pi$, $n=0,1,2,\ldots$ and in that case the Sturm-Liouville eigenfuctions are

$$\lambda=n^2\pi^2 \qquad \Longrightarrow \qquad \phi_n=\cos(n\pi x)$$

and the orthogonality property of the Sturm-Liouville eigenfunctions leads to

$$2\int_0^1\cos(n\pi x)\cos(m\pi x)\,dx=\delta_{m,n}$$

1.2. Bessel Functions. Taking

$$p(x) = x \quad , \quad q(x) = \frac{\nu^2}{x} \quad , \quad r(x) = -\frac{1}{x} \qquad , \qquad \lambda = n^2$$

leads to a Sturm-Liouville differential equation of the form

$$\frac{d}{dx}\left[x\frac{dy}{dx}\right] + \frac{\nu^2}{x}y - \frac{1}{x}n^2 y = 0$$

The solutions of the corresponding Sturm-Liouville problem when one imposes boundary conditions of the form

$$y(0) = 0$$
$$y(\infty) = 0$$

the (normalized) solutions are known as Bessel functions of the first kind, and are usually denoted by $J_n(x)$. From the orthogonality properties of Sturm-Liouville eigenfunctions one has

$$\int_0^\infty J_n(x)\, J_m(x)\, x dx = \delta_{n,m}$$

As we shall demonstrate below, Bessel functions arise naturally when one applies Separation of Variables to a Laplace operator expressed in polar coordinates.

Explicit expressions for the Bessel functions can be obtained several different ways. Using the method of genealized power series one can obtain solutions of the form

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(n+k)!}\left(\frac{x}{2\nu}\right)^{n+2k}$$

(There is also a second linear independent solution of the Bessel equation, but it diverges at $x = 0$).

1.3. Legendre Functions. Taking

$$p(x) = 1 - x^2 \quad , \quad q(x) = 0 \quad , \quad r(x) = 1$$
$$a = -1$$
$$b = 1$$

leads to Legendre's equation on the interval $[-1, 1]$

$$\left(\frac{d}{dx}\left(1 - x^2\right)\frac{dy}{dx}\right) + n(n+1)\, y = 0$$

This equation arises naturally when one applies Separation of Variables to the Laplace operator written in spherical coordinates (r, θ, ϕ). What happens is that the ODE corresponding to the angle θ (the longitudinal angle) takes the form

$$\frac{1}{\sin(\theta)}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) - \frac{m^2}{\sin^2(\theta)}\Theta + \lambda\Theta = 0$$

and then Legendre's equation pops out when one makes a change of variables $x = \cos\theta$.

1.4. Hermite Polynomials. Taking

$$p(x) = e^{-x^2/2} \quad , \quad q(x) = 0 \quad , \quad r(x) = e^{-x^2/2}$$
$$a = -\infty$$
$$b = +\infty$$

We are led to Hermite's differential equation

$$\frac{d}{dx}\left[e^{-x^2/2}\frac{dy}{dx}\right] = 2ne^{-x^2/y} = 0$$

It turns out the that solutions to this Sturm-Liouville problems constitute a set of *orthogonal polynomials*.

2. Example: Bessel's equation and Laplace's equation

Recall that in Problem Set 5, you were asked to write Laplace's equation

$$\nabla^2 \phi = 0$$

in polar coordinates and then separate variables to get a corresponding ODEs with respect to the polar coordinate variables r and θ. Let me review that computation quickly here.

Setting

$$\begin{array}{c} x = r\cos\theta \\ y = r\sin\theta \end{array} \quad \Longleftrightarrow \quad \begin{array}{c} r = \sqrt{x^2+y^2} \\ \theta = \tan^{-1}\left(\frac{y}{x}\right) \end{array}$$

and employing the change of variables formulas

$$\begin{aligned} \frac{\partial}{\partial x} &= \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial y} &= \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta} \end{aligned}$$

The Laplace's became

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = 0$$

Let me extend this computation to three dimensions adding in third variable z, representing the vertical direction (or at least a direction perpendicular to the xy-plane). Then under the change of variables

$$\begin{array}{c} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{array} \quad \Longleftrightarrow \quad \begin{array}{c} r = \sqrt{x^2+y^2} \\ \theta = \tan^{-1}\left(\frac{y}{x}\right) \\ z = z \end{array}$$

Laplace's equation becomes

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \tag{1}$$

Setting

$$\phi(r,\theta,z) = R(r)\,\Theta(\theta)\,Z(z) \tag{2}$$

and plugging that into (1) and then dividing by $R(r)\,\Theta(\theta)\,Z(z)$ yields

$$\frac{1}{R}R'' + \frac{1}{r}\frac{1}{R}R' + \frac{1}{r^2}\frac{1}{\Theta}\Theta'' + \frac{1}{Z}Z'' = 0$$

or

$$\frac{1}{R}R'' + \frac{1}{r}\frac{1}{R}R' + \frac{1}{r^2}\frac{1}{\Theta}\Theta'' = -\frac{1}{Z}Z''$$

Since the left hand side depends only on r and θ, while the right hand side depends only on z, both sides must be equal to a constant, which we'll choose to be $-\lambda^2$. We then have

$$\frac{1}{R}R'' + \frac{1}{r}\frac{1}{R}R' + \frac{1}{r^2}\frac{1}{\Theta}\Theta'' = -\lambda^2 \tag{3a}$$

$$-\frac{1}{Z}Z'' = -\lambda^2 \tag{3b}$$

Next, we multiply (3a) by r^2 and move the term involving θ to the right hand side and the term involving λ^2 to the left hand side. This yields

$$\frac{r^2}{R}R'' + \frac{r}{R}R' + \lambda^2 r^2 = -\frac{\Theta''}{\Theta}$$

The usual separation of variables argument now implies

$$\frac{r^2}{R}R'' + \frac{r}{R}R' + \lambda^2 r^2 = n^2 \tag{4a}$$

$$-\frac{\Theta''}{\Theta} = n^2 \tag{4b}$$

We note that the general solution of (4b)

$$\Theta(\theta) = c_1 e^{in\theta} + c_2 e^{-in\theta}$$

will not satisfy $\Theta(\theta) = \Theta(\theta + 2\pi)$ unless n is an integer.

Equations (3b), (4a), and (4b) can be rewritten as

$$r^2 R'' + rR' + \left(r^2\lambda^2 - n^2\right) R = 0 \tag{5a}$$

$$\Theta'' = -n^2\Theta \tag{5b}$$

$$Z'' = \lambda^2 Z \tag{5c}$$

Equations (5b) and (5c) are, respectively, the well-known ODEs for sine/cosine functions and exponential functions.

Note that after dividing (5a) by r we obtain

$$0 = rR'' + R' - \frac{n^2}{r}R + \lambda^2 rR = \frac{d}{dr}\left[r\frac{dR}{dr}\right] - \left(\frac{n^2}{r}\right)R + \lambda^2 rR$$

and that the far right hand side is of the form of a Sturm-Liouville differential equation

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] - q(x)\,y + \lambda r(x)\,y = 0 \qquad \text{with } p(x) > 0 \text{ and } r(x) > 0 \text{ for all } x \in [0,1].$$

if one takes

$$\begin{aligned} x &\longleftrightarrow r \\ p(x) &\longleftrightarrow r \\ q(x) &\longleftrightarrow \left(\frac{n^2}{r}\right) \\ r(x) &\longleftrightarrow r \\ \lambda &\longleftrightarrow \lambda^2 \end{aligned}$$

Equation (5a) is known as Bessel's equation. However, one usually writes Bessel's equation in a slightly different way. Introducing a change of variables,

$$x = \lambda r \quad \Longrightarrow \quad r = \frac{x}{\lambda} \quad \Longrightarrow \quad \frac{d}{dr} = \frac{dx}{dr}\frac{d}{dx} = \lambda\frac{d}{dx}$$

$$\begin{aligned} y(x) &= R(r(x)) = R\left(\frac{x}{\lambda}\right) \\ y'(x) &= \frac{1}{\lambda}R'\left(\frac{x}{\lambda}\right) \quad \Rightarrow \quad R'(r) = \lambda y'(x) \\ y''(x) &= \frac{1}{\lambda^2}R''\left(\frac{x}{\lambda}\right) \quad \Rightarrow \quad R''(r) = \lambda^2 y''(x) \end{aligned}$$

and so

$$r^2 R'' + rR' + \lambda^2 r^2 R - n^2 R = 0$$

becomes

$$\left(\frac{x}{\lambda}\right)^2 \lambda^2 y'' + \left(\frac{x}{\lambda}\right)\lambda y' + \lambda^2\left(\frac{x}{\lambda^2}\right) y - n^2 y = 0$$

or

$$x^2 y'' + xy' + \left(x^2 - n^2\right) y = 0 \tag{6}$$

or after dividing by x,

$$xy'' + y' + xy - \frac{n^2}{x} y = 0$$

which is the differential equation of the Bessel function of order n (as in § 17.1.2 above).

The case of the Bessel function is a nice representative example of how Sturm-Liouville problems arise in practice. In fact, this is especially true because when we get into the nuts and bolts of a phyical example (e.g., a vibrating circular membrane), we'll have to make a number of small adjustments to the procedures we used in Lectures 12 -13 to solve boundary value problems.

In fact, the first thing to work out is how does one actually solve an ODE of the general form

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + q(x)\,y + \lambda r(x)\,y = 0 \qquad ? \tag{7}$$

In the case, of Bessel's equation, we in fact have a regular singular point at the origin and so the Power Series technique taught in Math 2233 will have to be generalized. In the next couple of lectures, we'll review the Power Series technique and then extend it to a general method of tackling differential equations of the form (7).