LECTURE 16

Sturm-Liouville Theory and Nonhomogeneous BVPs

Recall from the last lecture the solution to a Sturm-Liouville problem is set of eigenvalues $\lambda_0, \lambda_1, \lambda_2, \dots$ and a corresponding set of functions $\phi_0(x)$, $\phi_1(x), \phi_2(x), \dots$ satisfying

$$(1) \qquad \frac{d}{dx}\left[p(x)\frac{d\phi_n}{dx}\right] - q(x)\phi_n + \lambda_n r(x)\phi_n = 0 \qquad , \qquad n = 0, 1, 2, \dots$$

and boundary conditions

$$(2) \qquad a_1\phi_n(0) + a_1\phi_n'(0) = 0 = b_1\phi_n(1) + b_2\phi_n'(1) \qquad , \qquad n = 0, 1, 2, \dots$$

Moreover, any continuous function $f : [0,1] \to \mathbb{R}$ can be expanded in terms of the S-L eigenfunctions $\{\phi_n \mid n \in \mathbb{N}\}$

$$(3) \qquad f(x) = \sum_{n=0}^{\infty} c_n\phi_n(x) \qquad , \qquad \text{with} \quad c_n := \int_0^1 f(x)\phi_n(x)r(x)\,dx \quad .$$

In this lecture, we consider the problem of developing a solution $\phi(x)$ of a related *nonhomogeneous* differential equation of the form

$$(4) \qquad \frac{d}{dx}\left[p(x)\frac{d\phi}{dx}\right] - q(x)\phi + \mu r(x)\phi = f(x)$$

satisfying the same boundary conditions

$$(5) \qquad a_1\phi(0) + a_1\phi'(0) = 0 = b_1\phi(1) + b_2\phi'(1)$$

We stress that the parameter μ need not be one of the Sturm-Liouville eigenvalues λ_n.

Suppose that $\phi(x)$ is a continuous solution of (4) and (5). Then by (3), we will have an expansion

$$(6) \qquad \phi(x) = \sum_{n=0}^{\infty} c_n\phi_n(x) \qquad , \qquad \text{with} \quad c_n := \int_0^1 \phi(x)\phi_n(x)r(x)\,dx$$

This expansion, in and of itself, is not yet very helpful, as we still have to know $\phi(x)$ in order to compute the coefficients c_n. However, if we insert the expansion (6) into the differential equation and use the fact that the S-L eigenfunctions $\phi_n(x)$ satisfy (1) and (2), we obtain

$$\begin{aligned}
f(x) &= \left[\frac{d}{dx}\left[p(x)\frac{d}{dx}\right] - q(x) + \mu r(x)\right]\left(\sum_{n=0}^{\infty} c_n\phi_n(x)\right) \\
&= \left(\sum_{n=0}^{\infty} c_n\phi_n\left[\frac{d}{dx}\left[p(x)\frac{d}{dx}\right] - q(x)\right]\phi_n(x)\right) + \sum_{n=0}^{\infty}\mu c_n r(x)\phi_n(x) \\
&= \left(\sum_{n=0}^{\infty} c_n\left(-\lambda_n r(x)\phi_n(x)\right)\right) + \sum_{n=0}^{\infty}\mu c_n r(x)\phi_n(x) \\
&= \left(\sum_{n=0}^{\infty} c_n(\mu - \lambda_n)r(x)\phi_n(x)\right)
\end{aligned}$$

Suppose we now multiply the extreme sides of this last equation by $\phi_m(x)$ and integrate from over the interval $[0,1]$

$$\begin{aligned}\int_0^1 f(x)\,\phi_m(x)\,dx &= \int_0^1 \left(\sum_{n=0}^{\infty} c_n(\mu-\lambda_n)\,r(x)\,\phi_n(x)\right)\phi_m(x)\,dx \\ &= \sum_{n=0}^{\infty} c_n(\mu-\lambda_n)\int_0^1 r(x)\,\phi_n(x)\,\phi_m(x)\,dx \\ &= \sum_{n=0}^{\infty} c_n(\mu-\lambda_n)\,\delta_{m,n} \\ &= c_m(\mu-\lambda_n)\end{aligned}$$

or

$$\int_0^1 f(x)\,\phi_m(x)\,dx = c_m(\mu-\lambda_n) \tag{7}$$

Case 1: $\mu \neq \lambda_n$. In this situation, the preceding equation allows us to immediately solve for the coefficients c_n of the S-L expansion (6) of the solution to (4), (5): viz.,

$$c_n = \frac{1}{\mu-\lambda_n}\int_0^1 f(x)\,\phi_n(x)\,dx \tag{8}$$

Case 2: $\mu = \lambda_n$, for some n. In this case, the condition (7) tells us that if

$$0 \neq \int_0^1 f(x)\,\phi_n(x)\,dx$$

then we'll have no solution of (7) and hence no solution of the original nonhomogeneous problem. On the other hand, if $\mu = \lambda_n$ and

$$0 = \int_0^1 f(x)\,\phi_n(x)\,dx$$

then condition (7) is vacuous. This means if we set

$$\begin{aligned}c_m &= \frac{1}{\mu-\lambda_m}\int_0^1 f(x)\,\phi_m(x)\,dx \qquad , \qquad m \neq n \\ c_n &= \text{arbitrary constant}\end{aligned}$$

then

$$\phi(x) = \sum_{m=0}^{\infty} c_m\phi_m(x)$$

will satisfy the nonhomogeneous differential equation (4) and boundary conditions (5).

Here is a theorem that summarizes the situation discussed above

THEOREM 16.1.

- *Let* $\{\lambda_0,\lambda_1,\lambda_2,\ldots\}$ *be the set of eigenvalues of a (homogeneous) Sturm-Liouville problem*

$$\begin{aligned}\frac{d}{dx}\left[p(x)\frac{d\phi}{dx}\right] - q(x)\,\phi + \lambda r(x)\,\phi &= 0 \\ a_1\phi(0) + a_1\phi'(0) &= 0 \\ b_1\phi(1) + b_2\phi'(1) &= 0\end{aligned}$$

and let $\{\phi_0,\phi_1,\ldots\}$ *be a corresponding set of Sturm-Liouville eigenfunctions, normalized so that*

$$\int_0^1 \phi_n(x)\,\phi_n(x)\,r(x)\,dx = 1 \quad .$$

The nonhomogeneous boundar value problem

$$\frac{d}{dx}\left[p\left(x\right)\frac{d\phi}{dx}\right]-q\left(x\right)\phi+\mu r\left(x\right)\phi=f\left(x\right)$$
$$a_1\phi\left(0\right)+a_1\phi'\left(0\right)=0$$
$$b_1\phi\left(1\right)+b_2\phi'\left(1\right)=0$$

has a unique solution whenever $\mu\notin\{\lambda_0,\lambda_1,\lambda_2,\ldots\}$ *; it is given by*

$$\phi\left(x\right)=\sum_{n=0}^{\infty}c_n\phi_n\left(x\right)$$

where$c_n=\frac{1}{\lambda_n-\mu}\int_0^1 f\left(x\right)\phi_n\left(x\right)dx$

- *If, on the other hand,* $\mu=\lambda_m$*, then the non-homogeneous problem has no solution unless*

(*) $$\int_0^1 f\left(x\right)\phi_m\left(x\right)dx=0$$

If, in fact, $\mu=\lambda_m$ *and (*) is true, then there is a one-parameter family of solutions given by*

$$\phi\left(x\right)=\sum_{n=0}^{\infty}c_n\phi_n\left(x\right)$$

where

$$c_n=\begin{cases}\frac{1}{\lambda_n-\mu}\int_0^1 f\left(x\right)\phi_n\left(x\right)dx & \text{if } n\neq m\\ \text{an arbitrary constant} & \text{if } n=m\end{cases}$$