LECTURE 14

Laplace's Equation

So far we've discussed the heat equation

$$\frac{\partial T}{\partial t} - \alpha^2 \nabla^2 T = 0 \quad , \tag{1}$$

and the wave equation

$$\frac{\partial^2 \phi}{\partial t^2} - \alpha^2 \nabla^2 \phi = 0 \quad . \tag{2}$$

The last prototypical PDE is Laplace's equation, which is

$$\nabla^2 \phi = 0 \quad . \tag{3}$$

Laplace's equation arises in a number of physical applications, one actually follows immediately from our discussion of the heat equation. Consider a system governed by the heat equation that is allowed to reach a time-independent state of equilibrium. In its equilibrium state we'll have

$$T(\mathbf{x}, t) = T_{ss}(\mathbf{x})$$

which will obey

$$0 = \frac{\partial T}{\partial t} - \alpha^2 \nabla^2 T = 0 - \alpha^2 \nabla^2 T_{ss} \quad \Longrightarrow \quad \nabla^2 T_{ss} = 0$$

1. Separation of Variables

In the following we'll consider the 2-dimensional Laplace equation

$$0 = \nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \tag{4}$$

and look for solutions of the form

$$\phi(x, y) = X(x) Y(y) \quad . \tag{5}$$

Plugging (5) into (4) and then dividing both sides by $X(x) Y(y)$ yields

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)}$$

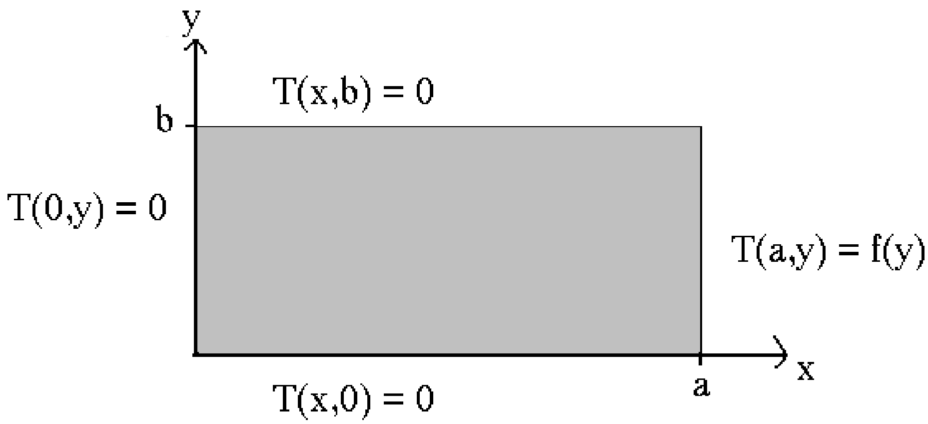
Applying the by now familiar separation-of-variables argument, we conclude that $X(x)$ and $Y(y)$ must satisfy equations of the form

$$X''(x) = CX(x) \tag{6a}$$

$$Y''(y) = -CY(x) \tag{6b}$$

2. Dirichlet Boundary Conditions

To make further progress towards a solution we'll now restrict attention to a particular physical situation with a particular set of boundary conditions. Consider a rectangular plate, three sides of which are immersed in a heat bath so that their temperatures are maintained at 0, and one side of which has its temperature maintained at prescribed function of y:



The boundary conditions require

(7a) $$0 = T(x,0) = X(x)Y(0) \quad \Longrightarrow \quad Y(0) = 0$$

(7b) $$0 = T(x,b) = X(x)Y(b) \quad \Longrightarrow \quad Y(b) = 0$$

(7c) $$0 = T(0,y) = X(0)Y(y) \quad \Longrightarrow \quad X(0) = 0$$

(7d) $$f(y) = T(a,y)$$

The boundary conditions on the right of (7a) and (7b) together with the differential equation (6b) require

$$C = \frac{n\pi}{b} \quad , \quad n = 1, 2, \ldots$$

and

$$Y(y) = \sin\left(\frac{n\pi}{b}y\right)$$

by an argument we have worked out several times before.

With $C = \frac{n\pi}{b}$ the general solution of (6a) will be

$$X(x) = c_1 \cosh\left(\frac{n\pi}{b}x\right) + c_2 \sinh\left(\frac{n\pi}{b}x\right)$$

which will satisfy the boundary condition (7c) only if we take $c_2 = 0$. If we now set

$$T(x,y) = \sum_{n=1}^{\infty} c_n \sinh\left(\frac{n\pi}{b}x\right) \sin\left(\frac{n\pi}{b}y\right)$$

then we have a solution not only of the PDE but also three out of the four boundary conditions. It remains to adjust the coefficients c_n so that the last boundary condition is satisfied

$$f(y) = T(a,y) \quad \Longrightarrow \quad f(y) = \sum_{n=1}^{\infty} c_n \sinh\left(\frac{n\pi}{b}a\right) \sin\left(\frac{n\pi}{b}y\right)$$

Employing the Fourier-sine expansion of $f(y)$

$$f(y) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{b}y\right) \qquad \text{with} \quad b_n = \frac{2}{b}\int_0^b f(x) \sin\left(\frac{n\pi}{b}y\right) dy$$

we can conclude that the solution of the PDE and boundary conditions is given by

$$T(x,y) = \sum_{n=1}^{\infty} c_n \sinh\left(\frac{n\pi}{b}x\right)\sin\left(\frac{n\pi}{b}y\right) \quad \text{with} \quad c_n = \frac{2}{b\sinh\left(\frac{n\pi a}{b}\right)}\int_0^b f(x)\sin\left(\frac{n\pi}{b}y\right)dy$$

3. Laplace's Equation on a Disk

Let's now consider Laplace's equation on a disk. This problem arises, for example, when one tries to figure out the steady state temperture distribution of a disk, when a prescribed temperature function is set around its perimeter.

$$\nabla^2\Phi(x,y) = 0 \tag{1}$$

$$\phi(R\cos\theta, R\sin\theta) = f(\theta) \tag{2}$$

Because of the circular symmetry of the disk, this problem is most easily solved by converting to polar coordinates

$$\begin{matrix} x = r\cos\theta \\ y = r\sin\theta \end{matrix} \quad \Longleftrightarrow \quad \begin{matrix} r = \sqrt{x^2+y^2} \\ \theta = \tan^{-1}\left(\frac{y}{x}\right) \end{matrix}$$

Using the two variable chain rule

$$\begin{aligned} \frac{\partial}{\partial x} &= \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial\theta}{\partial x}\frac{\partial}{\partial\theta} = \cos\theta\frac{\partial}{\partial r} - \frac{1}{r}\sin\theta\frac{\partial}{\partial\theta} \\ \frac{\partial}{\partial y} &= \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial\theta}{\partial y}\frac{\partial}{\partial\theta} = \sin\theta\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\frac{\partial}{\partial\theta} \end{aligned}$$

One finds (after a long calculation) that

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$$

Thus, Laplace's equation takes the form

$$\frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} = 0 \tag{3}$$

If we set $\Phi(r,\theta) = R(r)\Theta(\theta)$ and apply the Separation of Variables argument to (2), we find that the functions $R(r)$ and $\Theta(\theta)$ must satisfy

$$\begin{aligned} r^2R'' + rR' - \lambda^2 R &= 0 \\ \Theta'' + \lambda^2\Theta &= 0 \end{aligned} \tag{4}$$

where λ^2 is the "separation constant".

The solutions of (4) are relatively easy to find; however, they have different forms dependinng on whether or not $\lambda^2 = 0$.

$\lambda = 0$ For this situation,

$$\begin{aligned} \Theta'' = 0 \quad &\Rightarrow \quad \Theta(\theta) = A + B\theta \\ r^2R'' + rR' = 0 \quad &\Rightarrow \quad R(r) = C + D\ln|r| \end{aligned}$$

and so we get a separation of variables solution of the form

$$\Phi_0(r,\theta) = A_0 + B_0\ln|r| + C_0\theta + D_0\theta\ln\|r\|$$

with A_0, B_0, C_0 and D_0 arbitrary constants.

$\lambda \neq 0$ In this situation, we find

$$\begin{aligned} \Theta'' + \lambda^2 \Theta = 0 \quad &\Rightarrow \quad \Theta(\theta) = A\cos(\lambda\theta) + B\sin(\lambda\theta) \\ r^2 R'' + rR' - \lambda^2 R = 0 \quad &\Rightarrow \quad R(r) = Cr^{\lambda} + Dr^{-\lambda} \end{aligned}$$

and we get separation of variables solutions of the form

$$\Phi_\lambda(r,\theta) = A_\lambda \cos(\lambda\theta)\, r^{\lambda} + B_\lambda \sin(\lambda\theta)\, r^{\lambda} + C_\lambda \cos(\lambda\theta)\, r^{-\lambda} + D_\lambda \sin(\lambda\theta)\, r^{-\lambda} \quad .$$

Now any linear combination of these solutions

$$\begin{aligned} \Phi(r,\theta) &= A_0 + B_0 \ln|r| + C_0\theta + D_0\theta \ln\|r\| \\ &\quad + \sum_\lambda A_\lambda \cos(\lambda\theta)\, r^{\lambda} + \sum_\lambda B_\lambda \sin(\lambda\theta)\, r^{\lambda} + \sum_\lambda C_\lambda \cos(\lambda\theta)\, r^{-\lambda} + \sum_\lambda D_\lambda \sin(\lambda\theta)\, r^{-\lambda} \end{aligned} \tag{5}$$

will still be a solution of Laplace's equation. Our aim is to use the boundary conditions to fix a unique choice of coefficients $A_0, A_\lambda, \ldots$

Before imposing the stated boundary condition, we first note that there are two "hidden boundary" conditions based on our physical interpretation of the situation.

First of all, given the periodicity of the angular coordinate θ, we can demand

$$\Phi(r,\theta) = \Phi(r,\theta + 2\pi)$$

For this to happen, the λ's that occur on the right hand side of (5) must be integers; for

$$\begin{aligned} \cos(\lambda\theta) = \cos(\lambda(\theta + 2\pi)) \quad &\Rightarrow \quad \lambda \in \mathbb{Z} \\ \sin(\lambda\theta) = \sin(\lambda(\theta + 2\pi)) \quad &\Rightarrow \quad \lambda \in \mathbb{Z} \end{aligned}$$

We'll also need to set C_0 and D_0 equal to 0, since $F(\theta) = \theta$ is not periodic.

Secondly, we expect the temperature function $\Phi(r,\theta)$ to remain finite as $r \to 0$. This requires us to toss out the solutions involving factors of $\ln|r|$ and r^{-n}

We are thus left with the form the solution being

$$\Phi(r,\theta) = A_0 + \sum_{n=1}^{\infty} A_n \cos(n\theta)\, r^n + \sum_{n=1}^{\infty} B_n \sin(n\theta)\, r^n \tag{6}$$

Now we impose the boundary conditions (2)

$$f(\theta) = \Phi(R,\theta) = A_0 + \sum_{n=1}^{\infty} A_n \cos(n\theta)\, R^n + \sum_{n=1}^{\infty} B_n \sin(n\theta)\, R^n \tag{7}$$

Now as a function on the interval $[0, 2\pi]$, $f(\theta)$ has a Fourier expansion of the form

$$f(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\theta) + \sum_{n=1}^{\infty} b_n \sin(n\theta) \tag{8}$$

with coefficients determined by

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_0^{2\pi} f(\theta') \cos(n\theta')\, d\theta' \\ b_n &= \frac{1}{\pi} \int_0^{2\pi} f(\theta') \sin(n\theta')\, d\theta' \end{aligned}$$

Comparing the right hand sides of (7) and (8) and using the formulas for a_n and b_n we conclude

$$A_0 = \frac{1}{2\pi}\int_0^{2\pi} f(\theta')\,d\theta'$$
$$A_n = \frac{1}{R^n}\frac{1}{\pi}\int_0^{2\pi} f(\theta')\cos(n\theta')\,d\theta'$$
$$B_n = \frac{1}{R^n}\frac{1}{\pi}\int_0^{2\pi} f(\theta')\sin(n\theta')\,d\theta'$$

4. Poisson Sum Formula

Let me now write down, explicitly, our Separation of Variables solution to the Laplace's Equation on the Disk.

$$\begin{aligned}\Phi(r,\theta) &= A_0 + \sum_{n=1}^{\infty} A_n\cos(n\theta)\,r^n + \sum_{n=1}^{\infty} B_n\sin(n\theta)\,r^n \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(\theta')\,d\theta' + \sum_{n=1}^{\infty}\left(\frac{1}{\pi}\int_0^{2\pi} f(\theta')\cos(n\theta')\,d\theta\right)\cos(n\theta)\left(\frac{r}{R}\right)^n + \sum_{n=1}^{\infty}\left(\frac{1}{\pi}\int_0^{2\pi} f(\theta')\sin(n\theta')\,d\theta'\right)\sin(n\theta)\left(\frac{r}{R}\right)^n \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(\theta')\left(1 + 2\sum_{n=1}^{\infty}\left(\cos(n\theta)\cos(n\theta') + \sin(n\theta)\sin(n\theta')\right)\left(\frac{r}{R}\right)^n\right)d\theta\end{aligned}$$

or

$$\Phi(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} f(\theta')\left(1 + 2\sum_{n=1}^{\infty}\left(\cos(n\theta)\cos(n\theta') + \sin(n\theta)\sin(n\theta')\right)\left(\frac{r}{R}\right)^n\right)d\theta \tag{9}$$

Consider the expression inside the large parentheses

$$1 + 2\sum_{n=1}^{\infty}\left(\cos(n\theta)\cos(n\theta') + \sin(n\theta)\sin(n\theta')\right)\left(\frac{r}{R}\right)^n \tag{10}$$

Using the trig identity

$$\cos(A)\cos(B\ ) + \sin(A)\sin(B) = \cos(A-B)$$

We can rewrite (10) as

$$1 + 2\sum_{n=1}^{\infty}\cos(n\theta - n\theta')\left(\frac{r}{R}\right)^n \tag{11}$$

To simplify matters, let us temporarily set

$$\phi = \theta - \theta'$$
$$t = \frac{r}{R}$$

so that (11) becomres

$$1 + 2\sum_{n=1}^{\infty}\cos(n\phi)\,t^n \tag{12}$$

Now, via the Euler formular

$$\cos(n\phi) = \frac{e^{in\theta} + e^{-in\theta}}{2}$$

and so (12) can be written

$$1 + \sum_{n=1}^{\infty}\left(e^{in\theta}t^n + e^{-in\theta}t^n\right) = 1 + \sum_{n=1}^{\infty}\left(e^{i\phi}t\right)^n + \sum_{n=1}^{\infty}\left(e^{-\phi}t\right)^n \tag{(13)}$$

Now, by the Taylor formular for $\frac{1}{1-x}$,

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots = 1 + \sum_{n=1}^{\infty} z^n$$

we have

$$\sum_{n=1}^{\infty} \left(e^{i\phi}t\right)^n = \frac{1}{1-e^{i\phi}t} - 1 \quad , \quad \sum_{n=1}^{\infty} \left(e^{-i\phi}t\right)^n = \frac{1}{1-e^{-i\phi}t} - 1$$

And so (13) can be rewritten

$$\begin{aligned} 1 + \frac{1}{1-e^{i\phi}t} - 1 + \frac{1}{1-e^{\,-i\phi}\,t} - 1 &= -1 + \frac{1}{1-e^{i\phi}t} + \frac{1}{1-e^{-i\phi}t} \\ &= \frac{-\left(1-e^{i\phi}t\right)\left(1-e^{-i\phi}t\right) + \left(1-e^{i\phi}t\right) + \left(1-e^{-i\phi}t\right)}{\left(1-e^{i\phi}t\right)\left(1-e^{-i\phi}t\right)} \\ &= \frac{1-t^2}{1-te^{i\phi}-te^{-i\phi}+t^2} \\ &= \frac{1-t^2}{1-2t\cos(\phi)+t^2} \end{aligned}$$

Finally, putting everything back together we can conclude

$$\begin{aligned} 1 + 2\sum_{n=1}^{\infty} \left(\cos(n\theta)\cos(n\theta') + \sin(n\theta)\sin(n\theta')\right)\left(\frac{r}{R}\right)^n &= \frac{1-\left(\frac{r}{R}\right)^2}{1-2\left(\frac{r}{R}\right)\cos(\theta-\theta') + \left(\frac{r}{R}\right)^2} \\ &= \frac{R^2-r^2}{R^2-2rR\cos(\theta-\theta')+r^2} \end{aligned}$$

and so (9) can be written

$$\Phi(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} f(\theta')\left(\frac{R^2-r^2}{R^2-2rR\cos(\theta-\theta')+r^2}\right)d\theta \tag{14}$$

5. Geometric Expression of Solution

Let

$$\mathbf{x} = [r\cos(\theta), r\sin(\theta)]$$

be an arbitrary point in the disk, and let

$$\mathbf{y} = [R\cos(\theta'), R\sin(\theta')]$$

be an arbitrary point on the boundary of the disk. We have

$$\begin{aligned} \|\mathbf{x}-\mathbf{y}\|^2 &= \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 - 2\mathbf{x}\cdot\mathbf{y} \\ &= R^2 + r^2 - 2rR\cos(\theta-\theta') \end{aligned}$$

Thus, (14) can be expressed geometrically as

$$\begin{aligned} \Phi(r,\theta) &= \frac{1}{2\pi}\int_0^{2\pi} f(\mathbf{y})\left(\frac{\|\mathbf{y}\|^2-\|\mathbf{x}\|}{\|\mathbf{y}-\mathbf{x}\|^2}\right)d\theta \\ &= \frac{1}{2\pi R}\int_0^{2\pi} f(\mathbf{y})\left(\frac{\|\mathbf{y}\|^2-\|\mathbf{x}\|}{\|\mathbf{y}-\mathbf{x}\|^2}\right)Rd\theta \\ &= \frac{1}{2\pi R}\oint_{\partial D} f(\mathbf{y})\left(\frac{\|\mathbf{y}\|^2-\|\mathbf{x}\|}{\|\mathbf{y}-\mathbf{x}\|^2}\right)d\mathbf{y} \end{aligned}$$