LECTURE 13

The Wave Equation

The PDE that governs the propagation of (e.g. sound) waves in three dimensions is of the form

$$\frac{\partial^2 \phi}{\partial t^2} + k^2 \nabla^2 \phi = 0. \tag{1}$$

In fact, when the Laplace operator $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is generalized to the n-dimensional Laplace operator

$$\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$$

equation (1) corresponds to the proprogation of (if not the definition of) wave phenomena in an n-dimensional space. In this lecture we shall concentrate on the 1-dimensional wave equation

$$\frac{\partial^2 \phi}{\partial t^2} + k^2 \frac{\partial^2 \phi}{\partial x^2} = 0 \tag{2}$$

1. Separation of Variables

Once again our method of solution will be to

(i) Look first for solutions of the form $\phi(x,t) = X(x)\, Y(t)$
(ii) Utilize boundary conditions at $x = 0$ and $x = L$ to restrict choices of separation constants and solutions of the differential equations for $X(x)$ and $Y(t)$.
(iii) Construct a candidate solution for solutions satisfying a boundary condition at $t = 0$ out of the possibilities discovered in step (ii)
(iv) Impose the boundary condition at $t = 0$ to obtain a unique solution.

Thus, setting

$$\phi(x,t) = X(x)\, Y(t)$$

and inserting this into (1) yields[1]

$$X(x)\, \ddot{Y}(t) + k^2 Y(t)\, X''(x) = 0$$

Dividing both sides by $k^2 X(x)\, Y(t)$ yields

$$\frac{1}{k^2}\frac{\ddot{Y}(t)}{Y(t)} = \frac{X''(x)}{X(x)}$$

Since the left hand side depends only on t and the right hand side depends only on x and because this equation must hold for all x and t, we conclude that both sides must be equal to a constant, which we'll denote by C. Thus,

$$\frac{1}{k^2}\frac{\ddot{Y}(t)}{Y(t)} = C = \frac{X''(x)}{X(x)}$$

[1]Here we are employing the physicists' shorthand: $\ddot{Y} \equiv \frac{d^2 Y}{dt^2}$ and $X'' \equiv \frac{d^2 X}{dx^2}$

or

$$\ddot{Y} = \frac{C}{k^2}Y \tag{3}$$

$$X'' = CX \tag{4}$$

2. Boundary Conditions

If we think of (2) as the PDE governing the propagation of waves on a violin string, it is natural to impose the following boundary conditions of the form

$$\phi(0,t) = 0 \tag{5a}$$

$$\phi(L,t) = 0 \tag{5b}$$

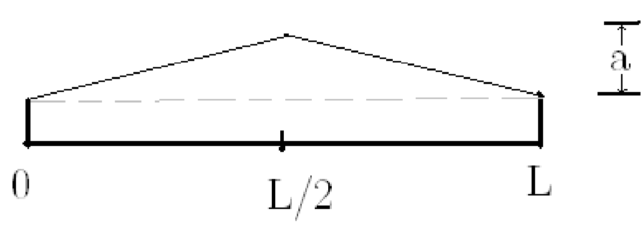
$$\phi(x,0) = f(x) \tag{5c}$$

$$\dot{\phi}(x,0) = g(x) \tag{5d}$$

For example, if we set

$$f(x) = \begin{cases} \frac{2a}{L}x & 0 \leq x \leq \frac{L}{2} \\ \frac{2a}{L}(L-x) & \frac{L}{2} \leq x < L \end{cases}$$
$$g(x) = 0$$

this would correspond to situation where the violin string of length L was released from an initial configuration shown below at time $t = 0$.



Now unless we choose the separation constant C to be negative, the solutions of (3) and (4) will either be exponentially decaying or decreasing:

$$y'' = Cy \quad \Longrightarrow \quad y = c_1 e^{\sqrt{C}x} + c_2 e^{-\sqrt{C}x}$$

Expecting instead oscillatory solutions we stipulate

$$C = -\lambda^2 < 0$$

This leads us to

$$\ddot{Y} = -\frac{\lambda^2}{k^2}Y \tag{6}$$

$$X'' = -\lambda^2 X \tag{7}$$

The general solution of (7) is

$$X(x) = A\sin(\lambda x + \delta)$$

Imposing the boundary conditions at $x = 0$ and $x = L$

$$0 = \phi(0,t) \quad \Longrightarrow \quad 0 = X(0)$$
$$0 = \phi(L,t) \quad \Longrightarrow \quad 0 = X(L)$$

requires us to take

$$\delta = 0$$
$$\lambda = \frac{n\pi}{L} \quad \text{for some } n \in \mathbb{N}$$

But then once we have $\lambda = n\pi/L$, the general solution of (6) is

$$Y(t) = c_1 \cos\left(\frac{n\pi}{kL}t\right) + c_2 \sin\left(\frac{n\pi}{kL}t\right)$$

Thus, functions of the form

$$\psi_{1,n} = \cos\left(\frac{n\pi}{kL}t\right) \sin\left(\frac{n\pi}{L}x\right) \quad , \quad n \in \mathbb{N}$$
$$\psi_{2,n} = \sin\left(\frac{n\pi}{kL}t\right) \sin\left(\frac{n\pi}{L}x\right) \quad , \quad n \in \mathbb{N}$$

will automatically satisfy the wave equation plus the boundary conditions at $x = 0$ and $x = L$

We now look for a linear combination of the functions $\psi_{1,n}$ and $\psi_{2,n}$ that will satisfy the boundary conditions at $t = 0$. Setting

$$\phi(x,t) = \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{kL}t\right) \sin\left(\frac{n\pi}{L}x\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{kL}t\right) \sin\left(\frac{n\pi}{L}x\right) \tag{8}$$

we have

$$f(x) = \phi(x,0) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi}{L}x\right) \tag{9}$$

$$g(x) = \dot{\phi}(x,0) = \sum_{n=1}^{\infty} \left(\frac{n\pi}{kL}\right) b_n \sin\left(\frac{n\pi}{L}x\right) \tag{10}$$

Comparing the right hand sides of (9) and (10) with the Fourier-sine expansions of, respectively, $f(x)$ and $g(x)$ we can conclude that if we choose

$$a_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi}{L}x\right) dx$$
$$b_n = \frac{2k}{n\pi} \int_0^L g(x) \sin\left(\frac{n\pi}{L}x\right) dx$$

then (8) will satisfy the wave equation (2) and the boundary conditions (5).