LECTURE 12

The Heat Equation

1. Derivation of the Heat Equation

Definition 12.1. ***Heat*** *is the energy transferred from one body to another due to a difference in temperature. (Better: heat is the kinetic energy of the molecules that compose the material.)*

There are two basic physical principles governing the notion of heat.

(i) The total heat energy H contained in a uniform, homogeneous body is related to its temperature T and mass in the following simple way

$$H = \kappa_s M T$$

where κ_s is the *specific heat capacity* of the material (a measurable constant specific to the material from with the body is made). More generally, in a situation for which neither the temperature nor the density of the material is constant we have

$$H(t) = \kappa_s \int_V \rho(\mathbf{x}) T(\mathbf{x}, t) \, d\mathbf{x} \tag{1}$$

(ii) The rate of heat transfer across a portion S of the boundary of a region R of the body is proportional directional derivative of T across the boundary and the area of contact

$$\text{Heat flux across } S = \sigma \int_S \nabla T \cdot \mathbf{n} dS \tag{2}$$

where $\mathbf{n} = \mathbf{n}(\mathbf{x})$ is the direction normal to the surface of contact at the point $\mathbf{x}$, and σ is another constant specific to the material from with the body is constructed. σ is called *heat conductivity constant*.

Applying Gauss's divergence theorem to (2) we have

$$\text{Heat flux entering/leaving a region} = \sigma \int_{\partial R} \nabla T \cdot \mathbf{n} dS = \sigma \int_R \nabla \cdot \nabla T \; d\mathbf{x} \tag{3}$$

This should be the (total) rate at which heat enters or leaves the region R, which in turn should correspond to the rate of change of the total amount of heat energy contained in the region:

$$\frac{dH}{dT} = \kappa_s \int_R \rho \frac{\partial T}{\partial t} d\mathbf{x} \tag{4}$$

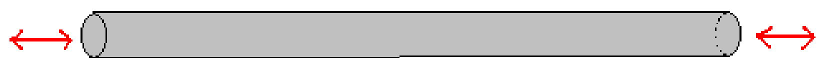
Equating (3) and (4) we thus obtain

$$\sigma \int_R \nabla \cdot \nabla T \; d\mathbf{x} = \kappa_s \int_R \rho \frac{\partial T}{\partial t} d\mathbf{x}$$

Since the region R can be chosen arbitrarily, the two integrands must coincide at every point of the body. We thus obtain

$$\nabla^2 T - \frac{\rho \kappa_s}{\sigma} \frac{\partial T}{\partial t} = 0 \tag{The Heat Equation}$$

1.1. The 1-dimensional Heat Equation. Above we derived the 3-dimensional heat equation. Let me now reduce the underlying PDE to a simpler subcase. Consider a long uniform tube surround by an insulating material like styroform along its length, so that heat can flow in and out only from its two ends:



In such a situation we can assume that the temperature really only depends on the position x along the length of the *heat pipe*. Then

$$\nabla^2 T \equiv \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \approx \frac{\partial T^2}{\partial x^2}$$

and the heat equation reduces to a 2-dimensional PDE of the form

$$\frac{\partial T}{\partial t} - \alpha^2 \frac{\partial^2 T}{\partial x^2} \tag{5}$$

where

$$\alpha = \sqrt{\frac{\sigma}{\rho\kappa_s}}$$

(Replacing the ratio $\sigma/(\rho\kappa_s)$ by α^2 will prove convenient later on.)

2. Separation of Variables

We begin by looking for solutions of (5), (6) and (6) of the form

$$T(x,t) = X(x)\, Y(t) \tag{8}$$

In other words, we look first for a solution which is the product of a function X depending only on the spatial variable x and a function Y that depends only the the time variable t. Plugging (8) into (5) yields

$$X\left(\frac{dY}{dt}\right) = \alpha^2 \left(\frac{d^2 X}{dx^2}\right) Y$$

Dividing both sides by $\alpha^2 X(x)\, Y(t)$ we obtain

$$\frac{1}{\alpha^2}\frac{1}{Y(t)}\frac{dY}{dt}(t) = \frac{1}{X(x)}\frac{d^2 X}{dx^2}(x) \tag{9}$$

Noting that the left hand side depends only on x, that the right hand side depends only on x, and that this equation must hold for all x and t we can conclude that both sides must be constants. (Since the right hand side does not depend on t the left hand side must be independent of t, and since the left hand side does not dependent on x the right hand side must be independent of x). If we denote by C the common constant equal to the left and right hand sides of (9) we have

$$\frac{1}{\alpha^2}\frac{1}{Y(t)}\frac{dY}{dt}(t) = C = \frac{1}{X(x)}\frac{d^2 X}{dx^2}(x)$$

$$\frac{dY}{dt} = C\alpha^2 Y \tag{10}$$

$$\frac{d^2 X}{dx^2} = CX \tag{11}$$

Thus, assuming (8) we can replace the PDE (5) with a pair of ordinary differential equations.

Noting that if C is positive, the general solution $Y(t) = Ae^{C\alpha^2 t}$ is (rapidly) unbounded as $t \to \infty$, we shall restrict our attention to solutions for which C is negative, and thus set

$$C = -\lambda^2$$

to make this, henceforth, more or less automatic.

In the examples that follow below we shall look for solutions of the 1-dimensional heat equation (5) subject to particular boundary conditions. One thing to note in these examples is exactly how and when the boundary conditions are employed. In stark contract to the solution of ODEs with boundary conditions, where boundary conditions are generally employed almost as an afterthought once the general solution has been found, the boundary conditions that we impose on the solutions of (5) will be essential an indensible part of our method of solution.

3. Example: Homogeneous Dirichlet Boundary Conditions

In order to demonstrate the basic procedure with as few complications as possible, I'll now specialize the PDE/Boundary Value Problem (5), (6), (8) to the case when both ends of the heat pipe are kept at the constant temperature 0. In what follows we shall continue to look for a solution such that $T(x,t) = X(x)Y(t)$. We thus demand

$$\frac{dY}{dt} = -\lambda^2\alpha^2 Y \tag{10}$$

$$\frac{d^2X}{dx^2} = -\lambda^2 X \tag{11}$$

and

$$f(x) = T(x,0) = X(x)Y(0) \tag{12a}$$

$$0 = T(0,t) = X(0)Y(t) \tag{12b}$$

$$0 = T(L,t) = X(L)Y(t) \tag{12c}$$

Ignoring for the moment the boundary condition (12a), we see that we can satisfy (12b) and (12c) by stipulating that

$$X(0) = 0 = X(L) \tag{13}$$

On the other hand, the general solution of

$$\frac{d^2X}{dx^2} = -\lambda^2 X$$

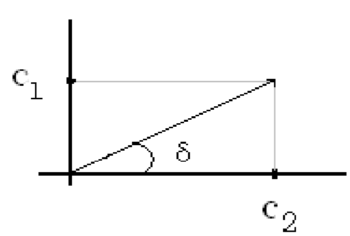
is

$$X(x) = c_1\cos(\lambda x) + c_2\sin(\lambda x)$$

We can also write this as

$$X(x) = \sqrt{c_1^2 + c_2^2}\sin\left(\lambda x + \tan^{-1}\left(\frac{c_1}{c_2}\right)\right)$$

via the following diagram



$$\delta = \tan^{-1}\left(\frac{c_1}{c2}\right) \ , \ A = \sqrt{c_1^2 + c_2^2} \quad \Longleftrightarrow \quad c_1 = A\sin\delta \ , \ c_2 = A\cos\delta$$

whence

$$\begin{aligned} X(x) &= A\sin(\delta)\cos(\lambda x) + A\cos\delta\sin(\lambda x) \\ &= A\sin(\lambda x + \delta) \end{aligned}$$

The easiest way to get nontrivial solutions to satisfy the boundary conditions (13) is to stipulate $\delta = 0$ and $\lambda = \frac{n\pi}{L}$, with $n \in \mathbb{N}$. But note also that if we do not choose δ and λ in this fashion, then the only way to satisfy the boundary conditions $X(0) = 0 = X(L)$ is to take $A = 0 \implies X(x) = 0$ for all x. In particular,

- the separation constant $-\lambda^2$ first appearing in (9) and (10) can not be completely arbitrary if the solution $T(x,t) = X(x)Y(t)$ is to satisfy the boundary conditions $T(x,0) = 0 = T(L,0)$.

Let us then set

$$-\lambda^2 = -\frac{n^2\pi^2}{L^2}$$

and proceed with the solution of (10).

$$\frac{dY}{dt} = -\lambda^2\alpha^2 Y = -\frac{\alpha^2 n^2\pi^2}{L^2}Y$$

This is just the differential equation of an exponential function, the general solution of which being

$$Y(t) = Ce^{-\left(\frac{\alpha n\pi}{L}\right)^2 t}$$

Thus,

$$T(x,t) = Ce^{-\left(\frac{\alpha n\pi}{L}\right)^2 t}\sin\left(\frac{n\pi}{L}x\right) \tag{14}$$

will be a solution of the 1-dimensional heat equation satisfying the boundary conditions $T(x,0) = 0 = T(L,0)$.

Unfortunately, the above solution is unlikely to satisfy the boundary condition at $t = 0$:

$$f(x) = T(x,0)$$

What saves the day here is that fact that (14) actually gives an infinite number of solutions of (5), (12b) and (12c); for we get a distinct solution for each $n = 0, 1, 2, 3, \ldots$.. Morever, since the PDE (5) is a linear PDE, given a set of solutions $\left\{\phi_n(x,t) = e^{-\left(\frac{\alpha n\pi}{L}\right)^2 t}\sin\left(\frac{n\pi}{L}x\right) \mid n = 0, 1, 2, \ldots\right\}$, we can create even more solutions by taking linear combinations of the solutions $\phi_n(x,t)$. We thus set

$$T(x,t) = \sum_{n=0}^{\infty} c_n e^{-\left(\frac{\alpha n\pi}{L}\right)^2 t}\sin\left(\frac{n\pi}{L}x\right)$$

Let's **now** impose the boundary condition at $t = 0$:

$$f(x) = T(x,0) = \sum_{n=0}^{\infty} c_n \sin\left(\frac{n\pi}{L}x\right)$$

Apparently, the coefficients c_n should be chosen so that they correspond to the Fourier-sine expansion of the function $f(x)$:

$$c_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)$$

We conclude that the solution of

$$\begin{aligned} \frac{\partial T}{\partial t} - \alpha^2 \frac{\partial^2 T}{\partial x^2} &= 0 \\ T(0,t) &= 0 \\ T(L,t) &= 0 \\ T(x,0) &= f(x) \end{aligned}$$

is given by

$$T(x,t) = \sum_{n=0}^{\infty} c_n e^{-\left(\frac{\alpha n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi}{L}x\right)$$

where

$$c_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)$$

4. Example: Non-homogeneous boundary conditions

Let us now consider the solution of the 1-dimensional heat equation

$$\frac{\partial T}{\partial t} - \alpha^2 \frac{\partial^2 T}{\partial x^2} = 0 \tag{15}$$

subject to non-homogenous boundary conditions

$$T(0,t) = T_1 \tag{16a}$$

$$T(L,t) = T_2 \tag{16b}$$

$$T(x,0) = f(x) \tag{16c}$$

which might correspond to a situation where a long rod with an initial temperature distribution $f(x)$ has its two ends inserted into two different heat baths that are maintained at different temperatures.

Since we expect that eventually as $t \to \infty$ the rod will eventually reach a steady state temperature distribution that is independent of time, we shall suppose that if

$$\text{for } t \text{ sufficiently large} \quad T(x,t) \approx T_{ss}(x)$$

where $T_{ss}(x)$ is the (as yet undetermined) final steady state temperature distribution. Since even for large t, $T(x,t)$ must still satisfy (15), (16a) and (16b), we have for sufficiently large t

$$0 = \frac{\partial T_{ss}}{\partial t} - \alpha^2 \frac{\partial^2 T_{ss}}{\partial x^2} \quad \Longrightarrow \quad \frac{d^2 T_{ss}}{dx^2} = 0 \tag{17}$$

and

$$T_{ss}(0) = T_1 \quad , \quad T_{ss}(L) = T_2 \tag{18}$$

The differential equation $\frac{d^2 T_{ss}}{dx^2} = 0$ implies T_{ss} is a linear fuction of x,

$$T_{ss}(x) = Ax + B$$

and the boundary conditions (18) require the constants A and B to be

$$B = T_1 \quad \text{and} \quad A = \frac{T_2 - T_1}{L}$$

Thus,

$$T_{ss}(x) = \frac{T_2 - T_1}{L}x + T_1 \tag{19}$$

Let us now define an auxiliary function $\tau(x,t)$ by

$$T(x,t) = T_{ss}(x) + \tau(x,t) \tag{20}$$

Evidently, $\tau(x,t)$ represents the discrepancy between the actual solution and the final steady state solution. Plugging the right hand side of (20) into equations (15) and (16) we find (noting again $\frac{\partial^2 T_{ss}}{\partial x^2} = 0 = \frac{\partial T_{ss}}{\partial t}$)

$$\frac{\partial \tau}{\partial t} - \alpha^2 \frac{\partial^2 \tau}{\partial x^2} = 0$$

and

$$T_1 = T(0,t) = T_{ss}(0) + \tau(0,t) = T_1 + \tau(0,t) \quad \Longrightarrow \quad \tau(0,t) = 0$$
$$T_2 = T(L,t) = T_{ss}(L) + \tau(L,t) = T_2 + \tau(L,t) \quad \Longrightarrow \quad \tau(L,t) = 0$$
$$f(x) = T(x,0) = T_{ss}(x) + \tau(x,0) = \frac{T_2 - T_1}{L}x + T_1 + \tau(x,0) \quad \Longrightarrow \tau(x,0) = f(x) - \frac{T_2 - T_1}{L}x - T_1$$

Thus, $\tau(x,t)$ satisfies

$$\begin{aligned} \frac{\partial \tau}{\partial t} - \alpha^2 \frac{\partial^2 \tau}{\partial x^2} &= 0 \\ \tau(0,t) &= 0 \\ \tau(L,t) &= 0 \\ \tau(x,0) &= F(x) \quad , \quad \text{where } F(x) = f(x) - \frac{T_2 - T_1}{L}x - T_1 \end{aligned}$$

in other words, a PDE/BVP of the form (5), (6), (7). We can thus conclude from the results of the last section that

$$\tau(x,t) = \sum_{n=0}^{\infty} c_n e^{-\left(\frac{\alpha n \pi}{L}\right)^2 t} \sin\left(\frac{n\pi}{L}x\right)$$

where

$$c_n = \frac{2}{L}\int_0^L F(x) \sin\left(\frac{n\pi}{L}x\right)$$

Hence, the solution of equations (15) and (16) is

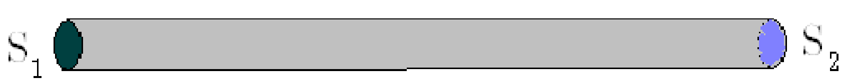
$$T(x,t) = \frac{T_2 - T_1}{L}x + T_1 + \sum_{n=0}^{\infty} c_n e^{-\left(\frac{\alpha n \pi}{L}\right)^2 t} \sin\left(\frac{n\pi}{L}x\right)$$

where

$$c_n = \frac{2}{L}\int_0^L \left(f(x) - \frac{T_2 - T_1}{L}x - T_1\right) \sin\left(\frac{n\pi}{L}x\right)$$

5. Example: homogeneous Neumann boundary conditions

Let us now consider a situation where a long rod is given an initial temperature distribution and then its ends are insulated so that no heat escapes. In this case, since no heat is flowing across the surfaces S_1 and S_2 at the end of the rod:



and we have from (2)

$$0 = \sigma \int_{S_1} \nabla T \cdot \mathbf{n} dS \quad \Longrightarrow \quad \frac{\partial T}{\partial x}(0,t) = 0$$
$$0 = \sigma \int_{S_2} \nabla T \cdot \mathbf{n} dS \quad \Longrightarrow \quad \frac{\partial T}{\partial x}(L,t) = 0$$

We thus look for solutions of

$$\frac{\partial T}{\partial t} - \alpha^2 \frac{\partial^2 T}{\partial x^2} = 0 \tag{21a}$$

$$\frac{\partial T}{\partial x}(0,t) = 0 \tag{21b}$$

$$\frac{\partial T}{\partial x}(L,t) = 0 \tag{21c}$$

$$T(x,0) = f(x) \tag{21d}$$

Our method will procede as in Section 3, up to the point where we imposed the boundary conditions (13). Thus, we begin by looking for solutions of the form

$$T(x,t) = X(x)\,Y(t)$$

which reduces the solution of equations (21a) to the solution of

$$X'' = -\lambda^2 X \tag{22a}$$

$$\dot{Y} = -\alpha^2\lambda^2 Y \tag{22b}$$

This time, however, we shall write the general solution of the ODE for X as

$$X(x) = A\cos(\lambda x + \delta)$$

The reason for using this (instead of the other choices $X(x) = c_1\cos(\lambda x) + c_2\sin(\lambda x)$ or $X(x) = A\sin(\lambda x + \delta)$) is that the boundary conditions at 0 and L

$$0 = X'(0) = -\lambda A\sin(\lambda 0 + \delta)$$
$$0 = X'(L) = -\lambda A\sin(\lambda L + \delta)$$

are quickly reduced to the requirements that

$$\delta = 0 \quad ; \quad \lambda = \frac{n\pi}{L} \quad , \quad n \in \mathbb{N} \tag{23}$$

whence

$$X(x) = c_n\cos\left(\frac{n\pi}{L}x\right) \quad , \quad n \in \mathbb{N}$$

With λ restricted as in (23) the general solution of (22b) is

$$Y(t) = Ae^{-\left(\frac{n\pi\alpha}{L}\right)^2 t}$$

and so the functions

$$\left\{\phi_n \equiv e^{-\left(\frac{n\pi\alpha}{L}\right)^2 t}\cos\left(\frac{n\pi}{L}x\right) \mid n \in \mathbb{N}\right\}$$

will provide us with an infinite set of independent solutions of (21a) - (21c). From these we now construct an ansatz for a function that will also satisfy (21d):

$$T(x,t) = \sum_{n=0}^{\infty} c_n e^{-\left(\frac{n\pi\alpha}{L}\right)^2 t}\cos\left(\frac{n\pi}{L}x\right) \tag{24}$$

Applying the boundary condition at $t = 0$

$$f(x) = T(x,0) = \sum_{n=0}^{\infty} c_n\cos\left(\frac{n\pi}{L}x\right)$$

we see that (24) will indeed satisfy (21a) - (21d) so long as the coefficients c_n are chosen to correspond with the Fourier-cosine coefficients of $f(x)$:

$$c_n = \frac{2}{L}\int_0^L \cos\left(\frac{n\pi}{L}x\right) f(x)\,dx \quad .$$