LECTURE 11

Fourier Series

A fundamental idea introduced in Math 2233 is that the solution set of a linear differential equation is a vector space. In fact, it is a vector subspace of a vector space of functions. The idea that functions can be thought of as vectors in a vector space is also crucial in what will transpire in the rest of this court.

However, it is important that when you think of functions as elements of a vector space V, you are thinking primarily of an abstract vector space - rather than a geometric rendition in terms of directed line segments. In the former, abstract point of view, you work with vectors by first adopting a basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ for V and then expressing the elements of V in terms of their coordinates with respect to that basis. For example, you can think about a polynomial

$$p = 1 + 2x + 3x^2 - 4x^3$$

as a vector, by using the monomials $\{1, x, x^2, x^3, \ldots\}$ as a basis and then thinking of the above expression for p as "an expression of p" in terms of the basis $\{1, x, x^2, x^3, \ldots\}$. But you can express p in terms of its Taylor series about $x = 1$:

$$p = 2 - 8(x-1) - 21(x-1)^2 - 16(x-1)^3 - 4(x-1)^4$$

and think of the polynomials $\left(1, x-1, (x-1)^2, (x-1)^3, \ldots\right)$ as providing another basis for the vector space of polynomials. Granted the second expression for p is uglier than the first, abstractly the two expressions are on an equal footing and moveover, in some situations the second expression might be more useful - for example, in understanding the behavior of p near $x = 1$. Indeed, the whole idea of Taylor series can be thought of as the means by which one expresses a given function in terms of a basis of the form $\left\{1, (x-x_0), (x-x_0)^2, \ldots\right\}$.

But there are many other interesting and useful bases for spaces of functions. The one we shall develop first is one that uses certain infinite families of trigonometric functions as a basis for the space of functions. A bit more explicitly, we shall consider functions of the form $\cos\left(\frac{m\pi}{L}x\right)$ and $\sin\left(\frac{m\pi}{L}x\right)$, where $m \in \mathbb{N}$ as a basis. As the utility of any basis is derived principally from the special properties of its members, the first thing we need do is discuss the special properities of these trigonometric functions.

1. Properties of Trignometric Functions

1.1. Periodicity.

Whenever a function f obeys a rule like

$$f(x+T) = f(x)$$

we say that f is *periodic* with *period* T. The key examples for what follows are the trigonometric functions $\cos(x)$ and $\sin(x)$; for which

$$\cos(x+2\pi) = \cos(\pi)$$
$$\sin(x+2\pi) = \sin(\pi)$$

which are periodic with period 2π. Moreover, for any integer n the functions $\cos(nx)$ and $\sin(nx)$ are also periodic with period 2π. For example, if $f = \cos(nx)$, $n \in \mathbb{Z}$, then

$$f(x+2\pi) = \cos(n(x+2\pi)) = \cos(nx + 2n\pi) = \cos(nx) = f(x).$$

Consider now the function $f(x) = \cos\left(\frac{\pi n}{L}x\right)$, $n = 0, 1, 2, \ldots$ We then have

$$f(x + 2L) = \cos\left(\frac{\pi n}{L}(x + 2L)\right) = \cos\left(\frac{\pi n}{L}x + 2n\pi\right) = \cos\left(\frac{\pi n}{L}x\right) = f(x)$$

Similarly, if $g(x) = \sin\left(\frac{\pi n}{L}x\right)$, $n = 0, 1, 2, , \ldots$ we have $g(x + 2L) = g(x)$.

Moreover, if we have any linear combination of functions of the form $\cos\left(\frac{\pi n}{L}x\right)$, $\sin\left(\frac{\pi n}{L}x\right)$, $n = 0, 1, 2, \ldots$

$$f(x) = \sum_n a_n \cos\left(\frac{\pi n}{L}x\right) + \sum_n b_n \sin\left(\frac{\pi n}{L}x\right)$$

we will have

$$f(x + 2L) = f(x)$$

And so the trigonometric functions $\cos\left(\frac{\pi n}{L}x\right)$, $\sin\left(\frac{\pi n}{L}x\right)$, $n = 0, 1, 2, \ldots$, provide a natural basis for constructing functions that are periodic with period $2L$.

1.2. Orthogonality. Recall that an inner product on a real vector space V is pairing $i : V \times V \longrightarrow \mathbb{R} : (u, v) \longrightarrow i(u, v)$ such that

- $i(v, u) = i(u, v)$ for all $u, v \in V$;
- $i(v, v) \geq 0$ for all $u \in V$; and
- $i(v, v) = 0 \quad \Longleftrightarrow \quad v = 0$.

Of course the prototypical inner product is the familar **dot product** for vectors in $\mathbb{R}^n$. There is also a natural inner product for the vector space of continuous functions with period $2L$.

$$(f, g) = \int_{-L}^{L}$$

Theorem 11.1. *Let V be the vector space of continuous functions on the interval $[-L, L] \subset \mathbb{R}$. Then the mappling*

$$f, g \longrightarrow \langle f, g\rangle := \int_{-L}^{L} f(x)\, g(x)\, dx$$

provides a positive-definite inner product on V. Moreover, if n, m are non-negative integers

$$\int_{-L}^{L} \cos\left(\frac{n\pi}{L}x\right) \cos\left(\frac{m\pi}{L}x\right) dx = \begin{cases} L & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$$
$$\int_{-L}^{L} \sin\left(\frac{n\pi}{L}x\right) \cos\left(\frac{m\pi}{L}x\right) dx = 0 \qquad \forall\ n, m \in \mathbb{N}$$
$$\int_{-L}^{L} \sin\left(\frac{n\pi}{L}x\right) \sin\left(\frac{m\pi}{L}x\right) dx = \begin{cases} L & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$$

Proof: (partial) By the addition and subtraction formulas for cosine functions

$$\cos(A + B) = \cos(A)\cos B - \sin(A)\sin(B)$$
$$\cos(A - B) = \cos(A)\cos(B) + \sin(A)\sin(B)$$

we have

$$\cos(A)\cos B = \frac{1}{2}\cos(A + B) + \frac{1}{2}\cos(A - B)$$

Thus, if $m \neq n$, then

$$\begin{aligned}
\int_{-L}^{L} \cos\left(\frac{n\pi}{L}x\right)\cos\left(\frac{m\pi}{L}x\right)dx &= \frac{1}{2}\int_{-L}^{L}\cos\left(\frac{\pi}{L}(n+m)x\right)dx + \frac{1}{2}\int_{-L}^{L}\cos\left(\frac{\pi}{L}(n-m)x\right)dx \\
&= \frac{1}{2}\left(\frac{-L}{\pi(n+m)}\sin\left(\frac{\pi}{L}(n+m)x\right)\right)\Bigg|_{-L}^{L} \\
&+ \frac{1}{2}\left(\frac{-L}{\pi(n-m)}\sin\left(\frac{\pi}{L}(n-m)x\right)\right)\Bigg|_{-L}^{L} \\
&= \frac{1}{2}\left(\frac{-L}{\pi(n+m)}\sin(\pi(n+m))\right) + \frac{1}{2}\left(\frac{L}{\pi(n+m)}\sin(\pi(n+m))\right) \\
&+ \frac{1}{2}\left(\frac{-L}{\pi(n-m)}\sin(\pi(n-m))\right) + \frac{1}{2}\left(\frac{L}{\pi(n-m)}\sin(\pi(n-m))\right) \\
&= 0+0+0+0
\end{aligned}$$

and if $m = n$

$$\begin{aligned}
\int_{-L}^{L} \cos\left(\frac{n\pi}{L}x\right)\cos\left(\frac{n\pi}{L}x\right)dx &= \frac{1}{2}\int_{-L}^{L}\cos\left(\frac{\pi}{L}(n+n)x\right)dx + \frac{1}{2}\int_{-L}^{L}\cos\left(\frac{\pi}{L}(n-n)x\right)dx \\
&= \frac{1}{2}\int_{-L}^{L}\cos\left(\frac{2n\pi}{L}x\right)dx + \frac{1}{2}\int_{-L}^{L}\cos(0)\,dx \\
&= \frac{1}{2}\left(\frac{-L}{2\pi n}\right)\sin\left(\frac{2\pi n}{L}x\right)\Bigg|_{-L}^{L} + \frac{1}{2}x\Bigg|_{-L}^{L} \\
&= 0+0+\frac{1}{2}L - \left(-\frac{1}{2}L\right) \\
&= L
\end{aligned}$$

2. Fourier Series

2.1. Definition.

DEFINITION 11.2. *A (formal) Fourier series is an expression of the form*

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right) \tag{1}$$

where $\{a_0, a_1, a_2, \ldots,\}$ *and* $\{b_1, b_2, b_3, \ldots\}$ *are sequences of real numbers.*

So long as the *coefficients* a_i and b_i tend to zero sufficiently fast, such series will converge to define a certain function of the parameter x. However, unlike power series, that is to say series of the form

$$g(x) = \sum_{n=0}^{\infty} c_n (x - x_o)^n \tag{2}$$

a Fourier series need not converge to a differentiable function, in fact, a Fourier series need not converge to a continuous function.We shall explore such phenomena a bit later.

Yet when a Fourier series does converge, it at least maintains the periodicity property of its component trigonometric functions; that is to say, if $f(x)$ is a convergent Fourier series then

$$f(x+L) = f(x) \quad .$$

2.2. Euler-Fourier Formula. If you know that a power series $g(x)$ as in (2) converges to a particular function, then it coincides with the Taylor expansion of $g(x)$ about x_o, and in fact the Taylor formula allows one to compute all of the coefficients c_n in terms of derivatives of $g(x)$

$$c_n = \frac{1}{n!} \left. \frac{d^n g}{dx^n} \right|_{x_o}$$

For Fourier series this is a somewhat analogous situation.

THEOREM 11.3. *Suppose*

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right)$$

is a convergent Fourier series. Then

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x) \cos\left(\frac{n\pi}{L}x\right) dx \tag{3a}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x) \sin\left(\frac{n\pi}{L}x\right) dx \tag{3b}$$

On the other hand, so long as $f(x)$ is an integrable function on the interval $[-L, L]$, then the formula (3a) and (3b) can be used to attach a particular Fourier series to $F(x)$:

$$f(x) \quad \to \left\{ \begin{array}{ll} a_n \equiv \frac{1}{L}\int_{-L}^{L} f(x) \cos\left(\frac{n\pi}{L}x\right) dx & n = 0, 1, 2, \ldots \\ b_n \equiv \frac{1}{L}\int_{-L}^{L} f(x) \sin\left(\frac{n\pi}{L}x\right) dx & n = 1, 2, \ldots \end{array} \right\} \quad \to \quad F(x) := \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right)$$

and it turns out that

THEOREM 11.4. *Suppose f and $\frac{df}{dx}$ are piece-wise continuous on the interval $[-L, L]$. Then f has a Fourier series expansion*

$$\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right) \\ a_n &\equiv \frac{1}{L}\int_{-L}^{L} f(x) \cos\left(\frac{n\pi}{L}x\right) dx \qquad n = 0, 1, 2, \ldots \\ b_n &\equiv \frac{1}{L}\int_{-L}^{L} f(x) \sin\left(\frac{n\pi}{L}x\right) dx \qquad n = 1, 2, \ldots \end{aligned}$$

that converges to $f(x)$ at all points $x \in [-L, L]$ where $f(x)$ is continuous and to $\frac{1}{2}(f(x_+) - f(x_-))$ at all points where $f(x)$ is discontinuous.

We call such a Fourier series, the *Fourier expansion* of $f(x)$. (The caveat "almost everywhere" can even be removed if $F(x)$ is continuous).

EXAMPLE 11.5. Consider the following function on $[-L, L]$ with discontinuities at $x = -L, 0, L$:

$$f(x) := \begin{cases} a & x = -L \\ 0 & -L < x < 0 \\ b & x = 0 \\ L & 0 < x < L \\ c & x = L \end{cases}$$

We have

$$
\begin{aligned}
a_0 &= \frac{1}{L}\int_{-L}^{L} f(x)\,dx = \frac{1}{L}\int_{0}^{L} L dx = L\\
a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}x\right)dx = \int_{0}^{L}\cos\left(\frac{n\pi}{L}x\right)dx = \frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\Big|_0^L = 0 \quad , \qquad n = 1,2,\ldots\\
b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi}{L}x\right)dx = \int_{0}^{L}\sin\left(\frac{n\pi}{L}x\right)dx = \frac{-L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Big|_0^L = 0 \quad , \qquad n = 1,2,\ldots
\end{aligned}
$$

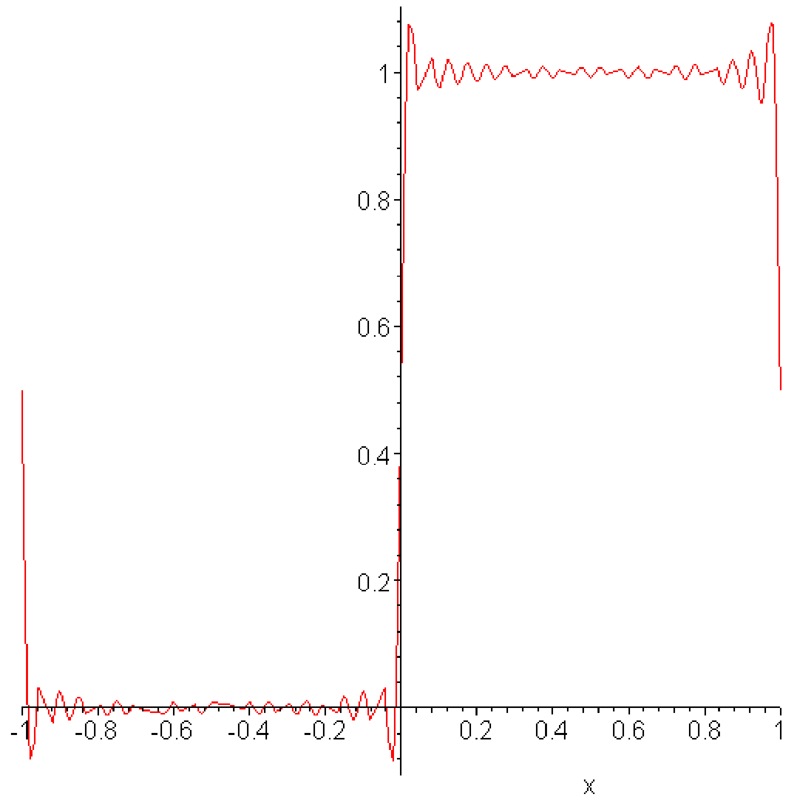
and for $n = 1,2,3,\ldots$

$$
\begin{aligned}
a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}x\right)dx = \int_{0}^{L}\cos\left(\frac{n\pi}{L}x\right)dx\\
&= \frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\Big|_0^L = 0\\
b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi}{L}x\right)dx = \int_{0}^{L}\sin\left(\frac{n\pi}{L}x\right)dx\\
&= \frac{-L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Big|_0^L = 0\\
&= \frac{L}{n\pi}\left(1-\cos(n\pi)\right)\\
&= \begin{cases} 0 & \text{if } n \text{ is even}\\ \frac{2L}{n\pi} & \text{if } n \text{ is odd}\end{cases}
\end{aligned}
$$

and so

$$
f(x) = \frac{L}{2} + \frac{2L}{\pi}\sum_{k=1}^{\infty}\frac{1}{2k-1}\sin\left(\frac{(2k-1)\pi}{L}x\right)
$$

Note that at $x = -L, 0, L$, the right hand side evaluates to $\frac{L}{2} = \frac{1}{2}\left(f(x_+) - f(x_-)\right)$. Below is a plot of the sum of the first 20 terms of the Fourier expansion of $f(x)$.



3. Fourier Sine and Cosine Series

The way we set things up the Fourier expansion of a function $f(x)$ that is continuous on an interval $[-L, L]$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right)$$

where

$$a_n := \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}x\right) dx$$
$$b_n := \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi}{L}x\right) dx$$

Suppose now that $f(x)$ is a function defined on the interval $[0, L]$. Then there are two simple ways of extending f to a function F on $[-L, L]$ and computing its Fourier expansion.

- Extend f to an *even* function F_{even} on $[-L, L]$ by setting

$$F_{even}(x) = \begin{cases} f(x) & 0 \le x \le L \\ f(-x) & -L \le x < 0 \end{cases}$$

- Extend f to an odd function F_{odd} on $[-L, L]$ by setting

$$F_{odd}(x) = \begin{cases} f(x) & 0 \le x \le L \\ -f(x) & -L \le x \le 0 \end{cases}$$

The Fourier coefficients of F_{even} will be

$$\begin{aligned}
a_n &= \int_{-L}^{L} F_{even}(x)\cos\left(\frac{n\pi}{L}x\right)dx \\
&= \frac{1}{L}\int_{-L}^{0} F_{even}(x)\cos\left(\frac{n\pi}{L}x\right)dx + \frac{1}{L}\int_{0}^{L} F_{even}(x)\cos\left(\frac{n\pi}{L}x\right)dx \\
&= -\frac{1}{L}\int_{L}^{0} F_{even}(-x')\cos\left(-\frac{n\pi}{L}x'\right)dx' + \frac{1}{L}\int_{0}^{L} F_{even}(x)\cos\left(\frac{n\pi}{L}x\right)dx \\
&= \frac{1}{L}\int_{0}^{L} f(x')\cos\left(\frac{n\pi}{L}x'\right)dx' + \frac{1}{L}\int_{0}^{L} f(x)\cos\left(\frac{n\pi}{L}x\right)dx \\
&= \frac{2}{L}\int_{0}^{L} f(x)\cos\left(\frac{n\pi}{L}x\right)dx
\end{aligned}$$

$$\begin{aligned}
b_n &= \int_{-L}^{L} F_{even}(x)\sin\left(\frac{n\pi}{L}x\right)dx \\
&= \frac{1}{L}\int_{-L}^{0} F_{even}(x)\sin\left(\frac{n\pi}{L}x\right)dx + \frac{1}{L}\int_{0}^{L} F_{even}(x)\sin\left(\frac{n\pi}{L}x\right)dx \\
&= -\frac{1}{L}\int_{L}^{0} F_{even}(-x')\sin\left(-\frac{n\pi}{L}x'\right)dx' + \frac{1}{L}\int_{0}^{L} F_{even}(x)\sin\left(\frac{n\pi}{L}x\right)dx \\
&= -\frac{1}{L}\int_{0}^{L} f(x')\sin\left(\frac{n\pi}{L}x'\right)dx' + \frac{1}{L}\int_{0}^{L} f(x)\cos\left(\frac{n\pi}{L}x\right)dx \\
&= 0
\end{aligned}$$

and so

$$F_{even}(x) = \frac{a_0}{2} + \sum_{n=0} a_n\cos\left(\frac{n\pi}{L}x\right) \quad ; \quad a_n = \frac{2}{L}\int_{0}^{L} f(x)\cos\left(\frac{n\pi}{L}x\right)dx$$

On the other hand, on the interval $[0, L]$, $F_{even}(x)$ must agree with the original function $f(x)$. Thus,

$$f(x) = \frac{a_0}{2} + \sum_{n=0} a_n\cos\left(\frac{n\pi}{L}x\right) \quad ; \quad a_n = \frac{2}{L}\int_{0}^{L} f(x)\cos\left(\frac{n\pi}{L}x\right)dx \qquad ; \qquad \forall\, x \in [0, L]$$

This expansion of $f(x)$, valid on an interval $[0, L]$ is called the Fourier-cosine expansion of $f(x)$.

Similarly, we can compute the Fourier expansion of $F_{odd}(x)$, and it turns out its Fourier coefficients are given by.

$$\begin{aligned}
a_n &= 0 \quad , \quad n = 0, 1, 2, 3, \ldots \\
b_n &= \int_{0}^{L} f(x)\sin\left(\frac{n\pi}{L}x\right)dx
\end{aligned}$$

Since $F_{odd}(x)$ must agree with $f(x)$ on $[0, L]$ we have

$$f(x) = \sum_{n=0} b_n\sin\left(\frac{n\pi}{L}x\right) \quad ; \quad b_n = \frac{2}{L}\int_{0}^{L} f(x)\sin\left(\frac{n\pi}{L}x\right)dx \qquad ; \qquad \forall\, x \in [0, L]$$

The right hand side is called the Fourier-sine expansion of $f(x)$.

In summary, a given function can be expanded in terms of trignometric functions several different ways:

(General Fourier series) $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right) \quad , \quad \forall\ x \in [L, L]$$

$$a_n := \frac{1}{L}\int_{-L}^{L} f(x) \cos\left(\frac{n\pi}{L}x\right) dx$$

$$b_n := \frac{1}{L}\int_{-L}^{L} f(x) \sin\left(\frac{n\pi}{L}x\right) dx$$

(Fourier-cosine series) $$f(x) = \frac{a_0}{2} + \sum_{n=0} a_n \cos\left(\frac{n\pi}{L}x\right) \quad ; \quad \forall\ x \in [0, L]$$

$$a_n = \frac{2}{L}\int_{0}^{L} f(x) \cos\left(\frac{n\pi}{L}x\right) dx$$

(Fourier-sine series) $$f(x) = \sum_{n=0} b_n \sin\left(\frac{n\pi}{L}x\right) \quad ; \quad \forall\ x \in [0, L]$$

$$b_n = \frac{2}{L}\int_{0}^{L} f(x) \sin\left(\frac{n\pi}{L}x\right) dx$$