LECTURE 3

Nonhomogeneous Linear Systems

We now turn our attention to nonhomogeneous linear systems of the form

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}(t)\,\mathbf{x}(t) + \mathbf{g}(t) \tag{1}$$

where $\mathbf{A}(t)$ is a (potentially t-dependent) matrix and $\mathbf{g}(t)$ is some prescribed vector function of t. As in the last lecture, we shall concentrate on 2×2 linear systems; as they are simple to compute and yet they still retain the essential features of the general case.

Just as in the case a single nonhomogenous linear ODE, the general solution of (1) will be of the form

$$\mathbf{x}(t) = \mathbf{x}_p(t) + \mathbf{x}_o(t)$$

where $\mathbf{x}_p(t)$ is a *particular* solution of (1) and $\mathbf{x}_o(t)$ is the general solution of the corresponding homogeneous equation

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}(t)\,\mathbf{x}(t) \quad . \tag{2}$$

(This afterall is a consequence of the linearity of the system, not the number of equations.) And so, just as in the case of a single ODE, we will need to know the general solution of homogeneous system (2) in order to solve the nonhomogeneous system (1).

1. Diagonalizable Systems with Constant Coefficients

Let's begin with the simple 2×2 system of the form

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}(t) + \mathbf{g}(t) \tag{3}$$

where $\mathbf{A}$ is a constant (i.e. t-independent) diagonalizable matrix. Let (r_1, ξ), and (r_2, η) be the eigenvalue/eigenvector pairs for $\mathbf{A}$. We then have

$$\mathbf{x}^{(1)}(t) = e^{r_1 t}\xi \qquad , \qquad \mathbf{x}^{(2)}(t) = e^{r_2 t}\eta$$

as two fundamental solutions of (2). Recall that if we form a matrix $\mathbf{C}$ by using the eigenvectors ξ and η as, respectively, the first and second columns then

$$\mathbf{C}^{-1}\mathbf{A}\mathbf{C} = \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix} \equiv \mathbf{D}$$

And so if we define

$$\mathbf{y} = \mathbf{C}^{-1}\mathbf{x} \qquad , \qquad \mathbf{h}(t) = \mathbf{C}^{-1}\mathbf{g}(t) \tag{4}$$

and multiply both sides of (3) from the left by $\mathbf{C}^{-1}$ we get

$$\mathbf{C}^{-1}\frac{d\mathbf{x}}{dt} = \frac{d}{dt}\left(\mathbf{C}^{-1}\mathbf{x}\right) = \frac{d\mathbf{y}}{dt}$$

$$\begin{aligned} \mathbf{C}^{-1}\left(\mathbf{A}\mathbf{x}(t) + \mathbf{g}(t)\right) &= \mathbf{C}^{-1}\mathbf{A}\mathbf{x}(t) + \mathbf{C}^{-1}\mathbf{g}(t) = \mathbf{C}^{-1}\mathbf{A}\mathbf{C}\mathbf{C}^{-1}\mathbf{x}(t) + \mathbf{C}^{-1}\mathbf{g}(t) \\ &= \mathbf{D}\mathbf{y}(t) + \mathbf{h}(t) \end{aligned}$$

Thus, we get

$$\frac{d\mathbf{y}}{dt} = \mathbf{D}\mathbf{y}(t) + \mathbf{h}(t) \tag{5}$$

or

$$\begin{bmatrix} \frac{dy_1}{dt} \\ \frac{dy_2}{dt} \end{bmatrix} = \begin{bmatrix} r_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_1(t) \end{bmatrix} + \begin{bmatrix} h_1(t) \\ h_2(t) \end{bmatrix}$$

or

$$\frac{dy_1}{dt} - r_1 y_1 = h_1(t) \tag{6a}$$

$$\frac{dy_2}{dt} - r_2 y_2 = h_2(t) \tag{6b}$$

Thus, the change of variable (4) allows us to convert the original system into an uncoupled pair of inhomogeneous ODEs.

Recall that the general solution of

$$y' + p(t)y = g(t) \quad \Longrightarrow \quad y = \frac{1}{\mu(t)}\int \mu(t)g(t) + \frac{C}{\mu(t)} \qquad \text{where} \quad \mu(t) = \exp\left[\int p(t)\,dt\right] \tag{7}$$

In the cases at hand

$$p(r) \quad \longrightarrow \quad \begin{cases} -r_1 & , \quad \text{a constant} \\ -r_2 & , \quad \text{a constant} \end{cases} \quad , \text{ a constant}$$

$$g(t) \quad \longrightarrow \quad \begin{cases} h_1(t) \\ h_2(t) \end{cases}$$

and accordingly the solutions of (6a) and (6b) are

$$y_1(t) = e^{r_1 t}\int e^{-r_1 t}h_1(t)\,dt + c_1 e^{r_1 t}$$

$$y_2(t) = e^{r_2 t}\int e^{-r_2 t}h_2(t)\,dt + c_2 e^{r_2 t}$$

Hence, we can write

$$\mathbf{y}(t) = \begin{bmatrix} e^{r_1 t}\int e^{-r_1 t}h_1(t)\,dt + c_1 e^{r_1 t} \\ e^{r_2 t}\int e^{-r_2 t}h_2(t)\,dt + c_2 e^{r_2 t} \end{bmatrix}$$

as the general solution of the auxiliary, decoupled system (5).

To recover the general solution of the original system, all we have to do is multiply the solution $\mathbf{y}(t)$ from the left by $\mathbf{C}$, as

$$\mathbf{x}(t) = \mathbf{C}\mathbf{C}^{-1}\mathbf{x}(t) = \mathbf{C}\left(\mathbf{C}^{-1}\mathbf{x}(t)\right) = \mathbf{C}\mathbf{y}(t) \tag{8}$$

2. General Diagonalizable Systems

Recall that the general solution of a single linear ODE

$$y' + p(x)y = g(x) \tag{9}$$

is given by

$$y = \frac{1}{\mu(x)}\int \mu(x)g(x)\,dx + \frac{C}{\mu(x)} \qquad , \qquad \mu(x) = \exp\left[\int p(x)\,dx\right] \tag{10}$$

Let me state this result a little differently. First, note that if $g(x) = 0$, then

$$y = \frac{C}{\mu(x)}$$

and so

$$\mu(x)^{-1}$$

is interpretable as a fundamental solution for the corresponding homogeneous problem

$$y' + p(x)\,y = 0.$$

Let me denote by $\psi(x)$ this fundamental solution:

$$\psi(x) \equiv \mu(x)^{-1} = \exp\left[-\int p(x)\right] dx$$

Then in terms of the fundamental solution $\psi(x)$ of the corresponding homogeneous problem, the general solution of nonhomogeous equation (9) is

$$y = \psi(x)\int \psi(x)^{-1} g(x)\,dx + C\psi(x) \tag{10$'$}$$

In fact, the general solution of a nonhomgeneous system

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}(t)\,\mathbf{x}(t) + \mathbf{g}(t) \tag{11}$$

can also be expressed in terms of the fundamental solutions of the corresponding homogeneous problem

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}(t)\,\mathbf{x}(t) \tag{12}$$

Suppose that the homogeneous system (12) has been solved in such a way that we can express its general solution in terms of a fundamental matrix $\mathbf{\Psi}(t)$[1]

$$\mathbf{x}_0(t) = \mathbf{\Psi}(t)\,\mathbf{c} \quad \Longrightarrow \quad \frac{d\mathbf{x}_0}{dt} = \mathbf{A}(t)\,\mathbf{x}_0(t)$$

I claim that

$$\mathbf{x}(t) \equiv \mathbf{\Psi}(t)\int \mathbf{\Psi}^{-1}(t)\,\mathbf{g}(t)\,dt + \mathbf{\Psi}(t)\,\mathbf{c} \tag{13}$$

is then the general solution of

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}(t)\,\mathbf{x}(t) + \mathbf{g}(t)$$

Before demonstrating this claim, however, note that (13) is a very straightforward generalization of the solution(10$'$) of a single nonhomogeneous linear ODE to a system of nonhomogeneous linear ODEs.

[1]In other words, suppose we have found n independent solutions $\psi^{(1)}(t),\ldots,\psi^{(n)}(t)$ of an $n\times n$ homogeous linear system $\frac{d\mathbf{x}_o}{dt} = \mathbf{A}(t)\,\mathbf{x}_o$ and have rewritten the right hand side of expression of the general solution as a linear combination of the fundamental solutions

$$\mathbf{x}_0(t) = c_1\psi^{(1)}(t) + \cdots + c_n\psi^{(n)}(t)$$

as a matrix product

$$\mathbf{x}_0(t) = \begin{bmatrix} | & & | \\ \psi^{(1)}(t) & \cdots & \psi^{(n)}(t) \\ | & & | \end{bmatrix}\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$
$$\equiv \mathbf{\Psi}(t)\,\mathbf{c}$$

As for the demonstration that (13) is a solution of 11, that's easy

$$
\begin{aligned}
\frac{d}{dt}\mathbf{x}(t) &= \frac{d}{dt}\left(\mathbf{\Psi}(t)\int \mathbf{\Psi}^{-1}(t)\,\mathbf{g}(t)\,dt + \mathbf{\Psi}(t)\,\mathbf{c}\right) \\
&= \left(\frac{d\mathbf{\Psi}}{dt}\right)\int \mathbf{\Psi}^{-1}(t)\,\mathbf{g}(t)\,dt + \mathbf{\Psi}(t)\left(\frac{d}{dt}\int \mathbf{\Psi}^{-1}(t)\,\mathbf{g}(t)\,dt\right) + \left(\frac{d}{dt}\mathbf{\Psi}(t)\right)\mathbf{c} \\
&= \left(\mathbf{A}(t)\,\mathbf{\Psi}(t)\right)\int \mathbf{\Psi}^{-1}(t)\,\mathbf{g}(t)\,dt + \mathbf{\Psi}(t)\left(\mathbf{\Psi}^{-1}(t)\,\mathbf{g}(t)\right) + \mathbf{A}(t)\,\mathbf{\Psi}(t)\,\mathbf{c} \\
&= \mathbf{A}(t)\left(\mathbf{\Psi}(t)\int \mathbf{\Psi}^{-1}(t)\,\mathbf{g}(t)\,dt + \mathbf{\Psi}(t)\,\mathbf{c}\right) + \mathbf{g}(t) \\
&= \mathbf{A}(t)\,\mathbf{x}(t) + \mathbf{g}(t)
\end{aligned}
$$

The step

$$
\mathbf{\Psi}(t)\left(\frac{d}{dt}\int \mathbf{\Psi}^{-1}(t)\,\mathbf{g}(t)\,dt\right) \quad \rightarrow \quad \mathbf{\Psi}(t)\left(\mathbf{\Psi}^{-1}(t)\,\mathbf{g}(t)\right)
$$

is just the application of the fundamental theorem of calculus.

Example 3.1.

$$
\begin{aligned}
\frac{dx_1}{dt} &= 2x_1 - x_2 + e^t \\
\frac{dx_2}{dt} &= 3x_1 - 2x_2 + e^{-t}
\end{aligned}
$$

This set of differential equations corresponds to an inhomogeneous system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{g}(t)$ with

$$
\mathbf{A} = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \qquad , \qquad \mathbf{g}(t) = \begin{bmatrix} e^t \\ e^{-t} \end{bmatrix}
$$

We shall first find the fundamental matrix for the corresponding homogeneous system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$. This means finding the eigenvalues r_1, r_2 of $\mathbf{A}$, the correponding eigenvectors ξ_1, ξ_2 and then forming the matrix

$$
\mathbf{\Psi}(t) = \begin{bmatrix} | & | \\ e^{r_1 t}\xi_1 & e^{r_2}\xi_2 \\ | & | \end{bmatrix}
$$

each column of which being a fundamental solution.

Now

$$
0 = \det(\mathbf{A} - \lambda\mathbf{I}) = \det\begin{bmatrix} 2-\lambda & -1 \\ 3 & -2-\lambda \end{bmatrix} = \lambda^2 - 4 + 3 = (\lambda - 1)(\lambda + 1) \quad \Longrightarrow \quad \lambda = 1, -1
$$

$$
\begin{aligned}
\lambda = 1 \quad &\Longrightarrow \quad NullSp(\mathbf{A} - \lambda\mathbf{I}) = NullSp\left(\begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix}\right) = NullSp\left(\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}\right) \\
&\Longrightarrow \quad \mathbf{v}_{\lambda=1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\lambda = -1 \quad &\Longrightarrow \quad NullSp(\mathbf{A} - \lambda\mathbf{I}) = NullSp\left(\begin{bmatrix} 3 & -1 \\ 3 & -1 \end{bmatrix}\right) = NullSp\left(\begin{bmatrix} 3 & -1 \\ 0 & 0 \end{bmatrix}\right) \\
&\Longrightarrow \quad \mathbf{v}_{\lambda=-1} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}
\end{aligned}
$$

and so when we can take

$$
\begin{aligned}
r_1 = 1 \quad &, \quad \xi_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
r_2 = -1 \quad &, \quad \xi_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}
\end{aligned}
$$

the fundamental matrix for the homogenous $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ is

$$\mathbf{\Psi}(t) = \begin{bmatrix} e^t & e^{-t} \\ e^t & 3e^{-t} \end{bmatrix}$$

Using the formula

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

we find

$$\mathbf{\Psi}(t)^{-1} = \frac{1}{e^t\left(3e^{-t}\right) - e^t e^{-t}} \begin{bmatrix} 3e^{-t} & -e^{-t} \\ -e^t & e^t \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 3e^{-t} & -e^{-t} \\ -e^t & e^t \end{bmatrix}$$

We can now plug into the formula

$$\mathbf{x}(t) = \mathbf{\Psi}(t) \int \mathbf{\Psi}^{-1}(t)\,\mathbf{g}(t)\,dt + \mathbf{\Psi}(t)\,\mathbf{c}$$

for the general solution of $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{g}(t)$. We have

$$\mathbf{\Psi}(t)\,\mathbf{c} = \begin{bmatrix} e^t & e^{-t} \\ e^t & 3e^{-t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} e^t c_1 + e^{-t} c_2 \\ e^t c_1 + 3e^{-t} c_2 \end{bmatrix}$$

and

$$\begin{aligned}
\int \mathbf{\Psi}^{-1}(t)\,\mathbf{g}(t) &= \int \frac{1}{2} \begin{bmatrix} 3e^{-t} & -e^{-t} \\ -e^t & e^t \end{bmatrix} \begin{bmatrix} e^t \\ e^{-t} \end{bmatrix} dt \\
&= \int \begin{bmatrix} \frac{3}{2} - \frac{1}{2}e^{-2t} \\ -\frac{1}{2}e^{2t} + \frac{1}{2} \end{bmatrix} dt \\
&\equiv \begin{bmatrix} \int \left(\frac{3}{2} - \frac{1}{2}e^{-2t}\right) dt \\ \int \left(-\frac{1}{2}e^{2t} + \frac{1}{2}\right) dt \end{bmatrix} \\
&= \begin{bmatrix} \frac{3}{2}t + \frac{1}{4}e^{-2t} \\ -\frac{1}{4}e^{2t} + \frac{1}{2}t \end{bmatrix}
\end{aligned}$$

And so

$$\begin{aligned}
\mathbf{\Psi}(t) \int \mathbf{\Psi}^{-1}(t)\,\mathbf{g}(t)\,dt &= \begin{bmatrix} e^t & e^{-t} \\ e^t & 3e^{-t} \end{bmatrix} \begin{bmatrix} \frac{3}{2}t + \frac{1}{4}e^{-2t} \\ -\frac{1}{4}e^{2t} + \frac{1}{2}t \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{4}e^t\left(6t + e^{-2t} - 1 + 2te^{-2t}\right) \\ \frac{1}{4}e^t\left(6t + e^{-2t} - 3 + 6te^{-2t}\right) \end{bmatrix}
\end{aligned}$$

Thus, finally we have

$$\begin{aligned}
\mathbf{x}(t) &= \mathbf{\Psi}(t) \int \mathbf{\Psi}^{-1}(t)\,\mathbf{g}(t)\,dt + \mathbf{\Psi}(t)\,\mathbf{c} \\
&= \begin{bmatrix} \frac{1}{4}e^t\left(6t + e^{-2t} - 1 + 2te^{-2t}\right) \\ \frac{1}{4}e^t\left(6t + e^{-2t} - 3 + 6te^{-2t}\right) \end{bmatrix} + \begin{bmatrix} e^t c_1 + e^{-t} c_2 \\ e^t c_1 + 3e^{-t} c_2 \end{bmatrix}
\end{aligned}$$

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