# Math 4233
# Homework Set 6

Problem 1. Find the first three non-zero terms in each of two linearly independent solutions of

$$xy'' + y' - y = 0$$

valid near $x = 0$.

Problem 2. A vibrating drum head obeys the following PDE

$$\frac{\partial^2 \phi}{\partial t^2} - c^2 \nabla^2 \phi = 0 \tag{1}$$

where $\nabla^2$ is the 2-dimensional Laplacian

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

in polar coordinates. In this problem, you'll use separation of variables to formulate a (fairly general) solution and then solve a particular boundary value problem.

(a) Use Separation of Variables to reduce the PDE to a set of three weakly coupled ODEs.

(b) Use periodicity with respect to the angular variable $\theta$ to put a restriction on one of the separation constants.

(c) Note that the radial factors $R_{n,\lambda}(r)$ of the Separation of Variables solutions have a regular singularity at $r = 0$. Make a change of variables $r \to \lambda x$ to recast the differential equation as a differential equation of the form

$$x^2 y'' + xy' + (x^2 - n^2)y = 0$$

This differential equation is called *the Bessel equation of order* $n$ and its solutions are usually denoted by $J_n(x)$.

(d). Show that the corresponding radial solutions $R_{n,\lambda}(r) = J_n(\lambda r)$ satisfy the orthogonality conditions

$$\int_0^b R_{n,\lambda}(r) R_{n,\lambda'}(r)\, r dr = 0 \qquad \text{if } \lambda \neq \lambda' \tag{3}$$

provided the functions $R_{n,\lambda}(r)$ satisfy a boundary condition of the form

$$\alpha R_{n,\lambda}(b) + \beta R_{n,\lambda}(b) = 0$$

(This is very similar to the Sturm-Liouville situation, but because we have only one boundary condition, there is an additional subtlety.)

(e) Use the Method of Frobenius (the generalized power series technique) to find the indicial equations and recursion relations for solutions of (2).

(f) Find an explicit formulas for the functions $R_{n,\lambda}(r)$, $\Theta_n(\theta)$ and $T_\lambda(t)$ corresponding to Separation of Variables solutions $\phi_{n,\lambda}(r,\theta,t) = R_{n,\lambda}(r)\,\Theta_n(\theta)\,T_\lambda(t)$ to (1) that are radially symmetric (i.e., $\theta$-independent) and regular at $r = 0$.

(g). Write down a formula for the solution of (1) satisfying

$$\begin{aligned}
\phi(b,\theta,t) &= 0 \qquad \text{for all } \theta \text{ and } t \\
\phi(r,\theta,0) &= f(r) \text{ for all } \theta \text{ and for } 0 \leq r \leq b \\
\frac{\partial \phi}{\partial t}(r,\theta,0) &= 0 \qquad \text{for all } \theta \text{ and for all } 0 \leq r \leq b
\end{aligned}$$

which corresponds to the drum head being held fixed at its perimeter, and which was initially at rest with the prescribed initial displacement. You do not have to explicitly compute the integrals that provide the coefficients of the series solution.