Math 4233
Homework Set 3

1. The equation of motion for a spring-mass system with damping is

$$m\frac{dx^2}{dt^2} + c\frac{dx}{dt} + kx = 0$$

where m, c, k are positive constants (mass, damping coefficient and spring force constant).

(a) Write this equation as a system of two first order equations for $u_1(t) = x(t)$ and $u_2(t) = \frac{du}{dt}$.

(b) Show that $u_1 = 0$, $u_2 = 0$ is a critical point and analyze the nature and stability of the critical point as a function of the parameters m, c and k.

2. Consider the linear system

$$\frac{dx}{dt} = a_{11}x + a_{12}y \quad , \quad \frac{dy}{dt} = a_{21}x + a_{22}y$$

where a_{11}, a_{12}, a_{21}, and a_{22} are real constants and the corresponding matrix is diagonalizable.

$$p = a_{11} + a_{22} = trace\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad , \quad q = a_{11}a_{22} - a_{12}a_{21} = \det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

(a) Show that the critical point $(0,0)$ is a node if $q > 0$ and $p^2 - 4q > 0$

(b) Show that the critical point $(0,0)$ is a saddle point if $q < 0$

(c) Show that the critical point $(0,0)$ is a spiral point if $p \neq 0$ and $p^2 - 4q < 0$

(d) Show that the critical point $(0,0)$ is a stable center point if $p = 0$ and $q > 0$.

Hint: Note first that in terms of the eigenvalues r_1, r_2 of the coefficient matrix, $p = r_1 + r_2$ and $r_1 r_2 = q$.

3. For each nonlinear system below, verify that $(0,0)$ is a critical point and that the system is locally linear about $(0,0)$. Discuss the stability of the critical point $(0,0)$ by examining the corresponding linear system.

(a)

$$\frac{dx}{dt} = x - y^2 \quad , \quad \frac{dy}{dt} = x - 2y + x^2$$

(b)

$$\frac{dx}{dt} = x + y^2 \quad , \quad \frac{dy}{dt} = x + y$$

4. For each of the following systems carry out the following steps.

(i) Identify the critical points.
(ii) For each critical point $\mathbf{c}$, identify the corresponding linear system. Write down the general solution of these linear systems and discuss the stability of the solutions near the critical solution $\mathbf{x}(t) = \mathbf{c}$.
(iii) Plot the direction field of the original system and discuss the evolution of the system for various initial conditions.

(a)

$$\frac{dx}{dt} = x(1 - x - y) \quad , \quad \frac{dy}{dt} = y(1.5 - y - x)$$

(b)

$$\frac{dx}{dt} = x(1 - 0.5y) \quad , \quad \frac{dy}{dt} = y(-0.25 + 0.5x)$$