Math 4233
Homework Set 2

1. Find the general solution of the following non-autonomous, homogeneous linear system

$$\frac{dx_1}{dt} = (t+1)\,x_1 + (t-1)\,x_2$$
$$\frac{dx_2}{dt} = (t-1)\,x_1 + (t+1)\,x_2$$

2.Find the general solution of the following inhomogeneous linear system

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} e^{-2t} \\ -2e^{t} \end{bmatrix}$$

3. The equation of motion for a spring-mass system with damping is

$$m\frac{dx^2}{dt^2} + c\frac{dx}{dt} + kx = 0$$

where m, c, k are positive constants (mass, damping coefficient and spring force constant).

(a) Write this equation as a system of two first order equations for $u_1(t) = x(t)$ and $u_2(t) = \frac{du}{dt}$.

(b) Show that $u_1 = 0$, $u_2 = 0$ is a critical point and analyze the nature and stability of the critical point as a function of the parameters m, c and k.

4. Consider the linear system

$$\frac{dx}{dt} = a_{11}x + a_{12}y \quad , \quad \frac{dy}{dt} = a_{21}x + a_{22}y$$

where a_{11}, a_{12}, a_{21}, and a_{22} are real constants and the corresponding matrix is diagonalizable.

$$p = a_{11} + a_{22} = trace\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad , \quad q = a_{11}a_{22} - a_{12}a_{21} = \det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

(a) Show that the critical point $(0, 0)$ is a node if $q > 0$ and $p^2 - 4q > 0$

(b) Show that the critical point $(0, 0)$ is a saddle point if $q < 0$

(c) Show that the critical point $(0, 0)$ is a spiral point if $p \neq 0$ and $p^2 - 4q < 0$

(d) Show that the critical point $(0, 0)$ is a stable center point if $p = 0$ and $q > 0$.

Hint: Note first that in terms of the eigenvalues r_1, r_2 of the coefficient matrix, $p = r_1 + r_2$ and $r_1 r_2 = q$.

5. For each nonlinear system below, verify that $(0, 0)$ is a critical point and that the system is locally linear about $(0, 0)$. Discuss the stability of the critical point $(0, 0)$ by examining the corresponding linear system.

(a)

$$\frac{dx}{dt} = x - y^2 \quad , \quad \frac{dy}{dt} = x - 2y + x^2$$

(b)

$$\frac{dx}{dt} = x + y^2 \qquad , \qquad \frac{dy}{dt} = x + y$$

6. For each of the following systems carry out the following steps.

(i) Identify the critical points.
(ii) For each critical point $\mathbf{c}$, identify the corresponding linear system. Write down the general solution of these linear systems and discuss the stability of the solutions near the critical solution $\mathbf{x}(t) = \mathbf{c}$.
(iii) Plot the direction field of the original system and discuss the evolution of the system for various initial conditions.

(a)

$$\frac{dx}{dt} = x\,(1 - x - y) \qquad , \qquad \frac{dy}{dt} = y\,(1.5 - y - x)$$

(b)

$$\frac{dx}{dt} = x\,(1 - 0.5y) \qquad , \qquad \frac{dy}{dt} = y\,(-0.25 + 0.5x)$$