MATH 4063-5023
Homework Set 7

1. Find the characteristic polynomials and minimal polynomials of the following matrices.

(a) $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

- The characteristic polynomial of this matrix is

$$p_{\mathbf{A}}(\lambda) = \det \begin{pmatrix} 2-\lambda & 0 & 0 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{pmatrix} = (2-\lambda)^3$$

The minimal polynomial will always be a factor of the characteristic polynomial; in fact, it will be the factor $m_{\mathbf{A}}(x)$ of smallest degree such $m_{\mathbf{A}}(A) = 0$. The possible factors of $p_{\mathbf{A}}(\lambda)$ are $(2-\lambda)$, $(2-\lambda)^2, (2-\lambda)^3$. We have

$$(2 \cdot \mathbf{I} - \lambda)|_{\lambda=\mathbf{A}} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} - \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \mathbf{0}$$

so in fact the factor of lowest degree is the minimal polynomial. Thus

$$m_{\mathbf{A}}(x) = 2 - x$$

(b) $\begin{pmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$

- The characteristic polynomial is again $p_{\mathbf{A}}(x) = (2-x)^3$, and we have the same initial possibilities for the minimal polynomial. We evaluate each of these possibilities at $x = \mathbf{A}$

$$\begin{aligned}
(2 \cdot 1 - x)|_{x=\mathbf{A}} &= \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \neq \mathbf{0} \\
(2 \cdot 1 - x)^2\Big|_{x=\mathbf{A}} &= \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \neq \mathbf{0} \\
(2 \cdot 1 - x)^3\Big|_{x=\mathbf{A}} &= \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \mathbf{0}
\end{aligned}$$

So the minimal polynomial is $(2-x)^3$

2. Find the eigenvalues of the following matrice, and then for each eigenvalue, find a basis for the corresponding eigenspace, and state the algebraic and geometric multiplicity of eigenvalue.

(a) $\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$

- First we find the eigenvalues. This is done by finding the roots of the characteristic polynomial:

$$0 = \det \begin{pmatrix} 1-\lambda & 2 \\ 0 & 3-\lambda \end{pmatrix} = (1-\lambda)(3-\lambda) \quad \Rightarrow \quad \lambda = 1, 3$$

To find the eigenvectors corresponding to an eigenvalue $\lambda = r$, we solve the homogeneous linear system $(\mathbf{A} - r\mathbf{I})\,\mathbf{x} = \mathbf{0}$. Any basis for this solution space will provide us with a set of linearly independent eigenvectors spanning the corresponding eigenspace.

$\lambda = 1$:

$$(\mathbf{A} - (1)\,\mathbf{I})\,\mathbf{x} = \mathbf{0} \quad \Longleftrightarrow \quad \begin{pmatrix} 1-1 & 2 \\ 0 & 3-1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

or

$$\begin{pmatrix} 0 & 2 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The coefficient matrix for this homogeneous linear system row reduces to the following matrix in Reduced Row Echelon Form

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \Rightarrow \quad \text{solution is } x_2 = 0$$

The component x_1 of the solution vector is left as a free parameter. Thus,

$$\mathbf{x} = \begin{pmatrix} x_1 \\ 0 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Thus, $[1, 0]$ will be a basis for the solution space, hence a basis for the $\lambda = 1$ eigenspace.

$\lambda = 3$:

$$(\mathbf{A} - (3)\,\mathbf{I})\,\mathbf{x} = \mathbf{0} \quad \Longleftrightarrow \quad \begin{pmatrix} 1-3 & 2 \\ 0 & 3-3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

It's pretty clear that the solution of this homogeneous linear system will have $x_1 = 0$, while x_2 will be a free parameter. Thus,

$$\mathbf{x} = \begin{pmatrix} 0 \\ x_2 \end{pmatrix} = x_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and so $[0, 1]$ will be a basis for the solution space, and so also a basis for the $\lambda = 3$ eigenspace.

(b) $\begin{pmatrix} 2 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 2 & 2 \end{pmatrix}$,

- In this (and subsequent) parts, I'll just summarize the calculations following the steps used in part (a).

$$\det \begin{pmatrix} 2-\lambda & 0 & 0 \\ 2 & 1-\lambda & 0 \\ 2 & 2 & 2-\lambda \end{pmatrix} = (1-\lambda)(2-\lambda)^2 \quad \Rightarrow \quad \lambda = 1, 2$$

$\lambda = 1$:

$$(\mathbf{A} - (1)\,\mathbf{I}) = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 2 & 2 & 1 \end{pmatrix} \xrightarrow{\text{row reduction}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{\text{solutions}} \left\{ \begin{array}{r} x_1 = 0 \\ x_2 = -\frac{1}{2}x_3 \end{array} \right.$$

$$\xrightarrow{\text{solution vector}} \mathbf{x} = x_3 \begin{pmatrix} 0 \\ -\frac{1}{2} \\ 1 \end{pmatrix} \xrightarrow{\text{basis for } \lambda = 1 \text{ eigenspace}} \begin{pmatrix} 0 \\ -\frac{1}{2} \\ 1 \end{pmatrix}$$

$\lambda = 2$:

$$(\mathbf{A} - (2)\,\mathbf{I}) = \begin{pmatrix} 0 & 0 & 0 \\ 2 & -1 & 0 \\ 2 & 2 & 0 \end{pmatrix} \xrightarrow{\text{row reduction}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{\text{solutions}} \left\{ \begin{array}{r} x_1 = 0 \\ x_2 = 0 \end{array} \right.$$

$$\xrightarrow{\text{solution vector}} \mathbf{x} = x_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \xrightarrow{\text{basis for } \lambda = 2 \text{ eigenspace}} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

(c) $\begin{pmatrix} -3 & 0 & 4 \\ 0 & -1 & 0 \\ -2 & 7 & 3 \end{pmatrix}$

- $$\det(\mathbf{A}-\lambda\mathbf{I}) = \det\begin{pmatrix} -3-\lambda & 0 & 4 \\ 0 & -1-\lambda & 0 \\ -2 & 7 & 3-\lambda \end{pmatrix} = -\lambda^3-\lambda^2+\lambda+1 = -(\lambda-1)(\lambda+1)^2 \quad \Rightarrow \quad \lambda = 1, -1$$

$\lambda = 1:$

$$(\mathbf{A}-(1)\mathbf{I}) = \begin{pmatrix} -4 & 0 & 4 \\ 0 & -2 & 0 \\ -2 & 7 & 2 \end{pmatrix} \xrightarrow{\text{row reduction}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{\text{solutions}} \left\{ \begin{array}{c} x_1 = x_3 \\ x_2 = 0 \end{array} \right.$$

$$\xrightarrow{\text{solution vector}} \mathbf{x} = x_3\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \xrightarrow{\text{basis for } \lambda = 1 \text{ eigenspace}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$\lambda = -1:$

$$(\mathbf{A}-(-1)\mathbf{I}) = \begin{pmatrix} -2 & 0 & 4 \\ 0 & 0 & 0 \\ -2 & 7 & 4 \end{pmatrix} \xrightarrow{\text{row reduction}} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{\text{solutions}} \left\{ \begin{array}{c} x_1 = 2x_3 \\ x_2 = 0 \end{array} \right.$$

$$\xrightarrow{\text{solution vector}} \mathbf{x} = x_3\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \xrightarrow{\text{basis for } \lambda = -1 \text{ eigenspace}} \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$

(d) $\begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}$ (you'll have to work over $\mathbb{C}$ for this one.)

- We proceed as before, except we allow complex numbers to creep into our solutions.

$$\det(\mathbf{A}-\lambda\mathbf{I}) = \det\begin{pmatrix} 1-\lambda & 2 \\ -1 & 3-\lambda \end{pmatrix} = \lambda^2 - 4\lambda + 5 = (\lambda - 2 + i)(\lambda - 2 - i) \quad \Rightarrow \quad \lambda = 2+i\ ,\ 2-i$$

$\lambda = 2+i$

$$(\mathbf{A}-(2+i)\mathbf{I}) = \begin{pmatrix} -1-i & 2 \\ -1 & 1-i \end{pmatrix} \xrightarrow{R_2 \to (1+i)R_2} \begin{pmatrix} -1-i & 2 \\ -1-i & 2 \end{pmatrix} \xrightarrow{R_2 \to R_2 - R_1} \begin{pmatrix} -1-i & 2 \\ 0 & 0 \end{pmatrix}$$

$$\Rightarrow \quad x_1 = \frac{2}{1+i}x_2 = \frac{2(1-i)}{(1+i)(1-i)}x_2 = (1-i)x_2$$

$$\Rightarrow \quad \mathbf{x} = x_2\begin{pmatrix} 1-i \\ 1 \end{pmatrix} \quad \Rightarrow \quad \text{basis for } \lambda = 2+i \text{ eigenspace is } \left\{\begin{bmatrix} 1-i \\ 1 \end{bmatrix}\right\}$$

$\lambda = 2-i$

$$(\mathbf{A}-(2+i)\mathbf{I}) = \begin{pmatrix} -1+i & 2 \\ -1 & 1+i \end{pmatrix} \xrightarrow{R_2 \to (1-i)R_2} \begin{pmatrix} -1+i & 2 \\ -1+i & 2 \end{pmatrix} \xrightarrow{R_2 \to R_2 - R_1} \begin{pmatrix} -1+i & 2 \\ 0 & 0 \end{pmatrix}$$

$$\Rightarrow \quad x_1 = \frac{2}{1-i}x_2 = \frac{2(1+i)}{(1+i)(1-i)}x_2 = (1+i)x_2$$

$$\Rightarrow \quad \mathbf{x} = x_2\begin{pmatrix} 1+i \\ 1 \end{pmatrix} \quad \Rightarrow \quad \text{basis for } \lambda = 2-i \text{ eigenspace is } \left\{\begin{bmatrix} 1+i \\ 1 \end{bmatrix}\right\}$$

3. Let V be a vector space with basis $\{v_1, \ldots, v_m\}$.Find a basis for $L(V, V)$.

- Let for each $i, j \in \{1, \ldots, m\}$ let $T_{ij} : V \to V$ be the linear transformation defined by

$$T_{ij}(a_1v_1 + \cdots + a_mv_m) = a_jv_i$$

This gives us m^2 different linear transformations. To demonstrate that this set of n^2 linear transformations actually constitutes a basis for $L(V,V)$, it suffices to show that a linear combination of them can produce the zero transformation only if all coefficients are equal to $0_{\mathbb{F}}$. Let

$$S=\sum_{i,j=1}^{m} b_{ij}T_{ij}$$

If S were the zero transformation it would have to vanish on each basis vector v_k. But

$$0=S\left(v_k\right)=\sum_{i,j=1}^{m} b_{ij}T_{ij}\left(v_k\right)=\sum_{i=1}^{m} b_{ik}v_i \quad \Rightarrow \quad b_{ik}=0 \qquad , \qquad k=1,\dots m$$

since the basis vectors v_i are linearly independent. Thus, a linear combination of the linear transformations T_{ij} can yield the zero transformation only if we take all the coefficients equal to zero. Hence, the T_{ij} are all linearly independent. Since there are m^2 of them (which is the dimension of $L\left(V,V\right)$), they comprise a basis for $L\left(V,V\right)$.

4. Suppose V is an n-dimensional vector space and $T:V\to V$ is an endomorphims of V with n linearly independent eigenvectors $v_1,\dots,v_n$ with eigenvalues $\xi_1,\dots,\xi_n$. Set

$$f\left(x\right)=\prod_{i=1}^{n}\left(x-\xi_i\right)$$

(a) Show $f\left(T\right)=0$.

- Because V is n-dimensional and the eigenvectors $v_1,\dots,v_n$ are stipulated to be linearly independent, the vectors $v_1,\dots,v_n$ will comprise a basis for V. Thus, any $v\in V$ can be uniquely written as

(1) $$v=a_1v_1+\cdots+a_nv_n \qquad , \qquad a_1,\dots,a_n\in\mathbb{F} \quad .$$

Now consider the operators

$$S_i=\left.\left(x-\xi_i\right)\right|_{x=T}=T-\xi_i\mathbf{1}.$$

We have

(2) $$S_iS_j=\left(T-\xi_i\mathbf{1}\right)\left(T-\xi_j\mathbf{1}\right)=T^2-\xi_iT-\xi_jT+\xi_i\xi_j\mathbf{1}=\left(T-\xi_j\mathbf{1}\right)\left(T-\xi_i\mathbf{1}\right)=S_jS_i$$

and so these operators all commute. Also, when S_i acts on the i^{th} eigenvector we have

(3) $$S_iv_i=\left(T-\xi_i\mathbf{1}\right)v_i=T\left(v_i\right)-\xi_iv_i=\xi_iv_i-\xi_iv_i=\mathbf{0}_V \quad .$$

Using (3) and (2) we then have

$$f\left(T\right)v_i=\prod_{j=1}^{n}\left(T-\xi_j\mathbf{1}\right)v_i=S_1\cdots S_i\cdots S_nv_i=S_1\cdots S_{i-1}S_{i+1}\cdots S_nS_iv_i=S_1\cdots S_{i-1}S_{i+1}\cdots S_n\mathbf{0}_V=\mathbf{0}_V$$

So $f\left(T\right)$ vanishes on each basis vector v_i. But we can express every vector v in the form (1), we have

$$f\left(T\right)v=f\left(T\right)\left(a_1v_1+\cdots+a_nv_n\right)=a_1f\left(T\right)v_1+\cdots+a_nf\left(T\right)v_n=\mathbf{0}_V+\cdots+\mathbf{0}_V=\mathbf{0}_V \quad .$$

□

(b) Show the minimal polynomial of T is the product of the distinct factors $\left(x-\xi_i\right)$ of f.

- Notice that if $\xi_i=\xi_j$, one doesn't need both factors S_i and S_j to produce an operator that vanishes on both v_i and v_j via the argument of equation (3) in part (a). Therefore, if we use only the S_i corresponding to distinct eigenvalues, we'll still have an operator that is a polynomial in T

and vanishes on all of V. What we need to show is that the operator of lower degree. But we also know that the minimal polynomials has to divide

$$\widetilde{f}(x) = \prod_{\xi_i \neq \xi_j} (x - \xi_i)$$

since $f(T) = 0$ on V. But $\widetilde{f}(x)$ is already factored into irreducible polynomials, so any polynomial that divides $\widetilde{f}(x)$ must be a product of the same irreducibles, except perhaps missing some factors. However, if you remove a factor, say $(x - \xi_i)$ from $\widetilde{f}(x)$, you end up with a polynomial whose corresponding operator no longer vanishes on the ξ_i-eigenspace:

$$(T - \xi_1) \cdots (T - \xi_{i-1}) (T - \xi_{i+1}) \cdots (T - \xi_n) v_i = (\xi_i - \xi_1) \cdots (\xi_i - \xi_{i-1}) (\xi_i - \xi_{i+1}) \cdots (\xi_i - \xi_n) v_i \neq 0 \qquad \text{because } \xi_i \neq \xi_j$$

Thus, $\widetilde{f}(x)$ is the minimal polynomial. □

5. Show that $T \in L(V, V)$ is invertible if and only if the constant term of the minimal polynomial is not equal to zero. Come up with an algorithm for computing T^{-1} from its minimal polynomial.

- Write

$$m_T(x) = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1} + x^n$$

for the minimal polynomial. By definition, $m_T(x)$ is the monic polynomial of lowest degree such that

$$m_T(T) v = 0 \qquad \forall\, v \in V \quad .$$

$\Rightarrow$ Suppose T is invertible. Then the null space of T is $\{\mathbf{0}_V\}$ and the range of T is the entire space V. On the other hand, we have

(1) $$\mathbf{0}_V = m_T\ (T) v \quad \Rightarrow \quad a_0 v = -\left(a_1 + a_2 T + \cdots + a_n T^{n-1}\right) T v$$

Now if the constant term a_0 equals $0_{\mathbb{F}}$, then we have

$$-\left(a_1 + a_2 T + \cdots + a_n T^{n-1}\right) T(v) = \mathbf{0}_V \qquad \forall\, v \in V$$

But since the range of T is all of V, this requires

$$-\left(a_1 + a_2 T + \cdots + a_n T^{n-1}\right) v = 0 \qquad \forall\, v \in V \quad .$$

But this can't be, since $-\left(a_1 + a_2 T + \cdots + a_n T^{n-1}\right)$ is a polynomial in T of lower degree than the minimal polynomials. So we can conclude that T being invertible requires the constant term of the minimal polynomial for T to be non-zero.

$\Longleftarrow$ Suppose $a_0 \neq 0$. We whant to show that T is invertible. Equation (1) still holds, and can be rewritten as

$$v = \frac{1}{a_0}\left(a_1 + a_2 T + \cdots + a_n T^{n-1}\right) T v \qquad \forall\, v \in V$$

So

$$\frac{1}{a_0}\left(a_1 + a_2 T + \cdots + a_n T^{n-1}\right) T = \mathbf{I}_{L(V,V)}$$

and so

$$T^{-1} = \frac{1}{a_0}\left(a_1 + a_2 T + \cdots + a_n T^{n-1}\right) \quad .$$

Hence, T is invertible.

6. Two matrices $\mathbf{A}$ and $\mathbf{B}$ are said to be *similar* if there exists an invertible matrix $\mathbf{C}$ such that $\mathbf{B} = \mathbf{C}^{-1}\mathbf{A}\mathbf{C}$. Show that similar matrices have the same eigenvalues.

- The eigenvalues a matrix $\mathbf{A}$ are the roots of the equation $\det(\mathbf{A} - x\mathbf{I}) = 0$. Suppose $\mathbf{B} = \mathbf{C}^{-1}\mathbf{A}\mathbf{C}$. Then the eigenvalues of $\mathbf{B}$ will be the roots of

$$\begin{aligned} 0 &= \det\left(\mathbf{C}^{-1}\mathbf{A}\mathbf{C} - x\mathbf{I}\right) \\ &= \det\left(\mathbf{C}^{-1}\mathbf{A}\mathbf{C} - x\mathbf{C}^{-1}\mathbf{C}\right) \\ &= \det\left(\mathbf{C}^{-1}(\mathbf{A} - x\mathbf{I})\mathbf{C}\right) \\ &= \det\left(\mathbf{C}^{-1}\right)\det(\mathbf{A} - x\mathbf{I})\det(\mathbf{C}) \\ &= \frac{1}{\det(\mathbf{C})}\det(\mathbf{A} - x\mathbf{I})\det(\mathbf{C}) \\ &= \det(\mathbf{A} - x\mathbf{I}) \end{aligned}$$

  Since $\mathbf{A}$ and $\mathbf{B}$ have the same characteristic polynomial, their eigenvalues must coincide.

7. Suppose $T \in L(V, V)$ has $n = \dim V$ distinct eigenvalues. Show that there exists a basis of V consisting of eigenvectors of T. What will be the matrix of T with respect to this basis?

- The eigenvectors for distinct eigenvalues are linearly independent (see Theorem 16.11 of Lecture 16), the fact that we have as many distinct eigenvectors as the dimenion of V implies that the corresponding eigenvectors will provide a basis $B = \{v_1, \ldots, v_n\}$ for V. Let's write down the matrix T_{BB} of T corresponding to adapting the basis B for both the domain and codomain of the linear transformation $T : V \to V$. We have

$$T(v_i) = \lambda_i v_i \qquad \Rightarrow \qquad \text{the coordinate vector of } T(v_i) \text{ with respect to } B \text{ is } [0, 0, \ldots, 0, \lambda_i, 0, \ldots, 0]$$

  where the non-zero entry occurs precisely in the i^{th} slot. Thus, the matrix $\mathbf{T}_{BB}$ corresponding to the linear transformation T and and basis B will have the form

$$\mathbf{T}_{BB} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_2 & & \vdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & 0 & \lambda_{n-1} & 0 \\ 0 & \cdots & \cdots & 0 & \lambda_n \end{pmatrix}$$

  that is to say, $\mathbf{T}_{BB}$ will be a diagonal matrix whose diagonal entries are just the eigenvalues of T. $\square$