MATH 4063-5023
Homework Set 6

1. Let $B_1 = \{[1,1],[1,-1]\}$ and let $B_2 = \{[3,7],[-1,-3]\}$. Regarding B_1 and B_2 as bases for $\mathbb{R}^2$, find the change-of-coordinates-matrix that converts coordinate vectors with respect to B_1 to coordinate vectors w.r.t. B_2.

- Let $B_0 = \{[1,0],[0,1]\}$ be the standard basis for $\mathbb{R}^2$. Using the vectors of B_1 as columns, we can form the matrix that converts coordinate vectors w.r.t. to B_1 to coordinate vectors w.r.t. the standard basis.

$$C_{B_1B_0} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Similarly,

$$C_{B_2B_0} = \begin{bmatrix} 3 & -1 \\ 7 & -3 \end{bmatrix}$$

will convert coordinate vectors w.r.t. B_2 to coordinate vectors w.r.t. the standard basis B_0. The matrix inverse of $C_{B_2B_0}$ will then convert coordinate vectors w.r.t. B_0 to coordinate vectors w.r.t. B_2. An easy computation yields

$$(C_{B_2B_0})^{-1} = \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ \frac{7}{2} & -\frac{3}{2} \end{bmatrix}$$

We can now go from B_1 to B_0 to B_2 by multiplying $C_{B_1B_0}$ from the left by $(C_{B_2B_0})^{-1}$

$$C_{B_1B_2} = (C_{B_2B_0})^{-1} C_{B_1B_0} = \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ \frac{7}{2} & -\frac{3}{2} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$$

2. Let $B_1 = \{1, x, x^2\}$ and let $B_2 = \left\{1, x-1, (x-1)^2\right\}$. Regarding B_1 and B_2 as bases for the vector space of polynomials of degree ≤ 2, find the change-of-coordinates-matrix that converts coordinate vectors with respect to B_1 to coordinate vectors with respect to B_2.

- Let $p = a_0 + a_1x + a_2x^2$ be an arbitary polynomial. We are looking for a means of going from its coordinate $[a_0, a_1, a_2]$ w.r.t. the basis $\{1, x, x^2\}$ to a coordinate vector w.r.t to the basis of a polynomial $a_0 + a_1x + a_2x^2$ a coordinate vector w.r.t..the basis $\left\{1, x-1, (x-1)^2\right\}$. In other words, we need to solve a equation like

$$a_1 + a_2x + a_3x^2 = b_1 + b_2(x-1) + b_3(x-1)^2$$

expressing b_1, b_2, b_3 in terms of a_1, a_2, a_3..

But we can also do this by constructing the change of basis matrix directly. Let me set $e_1 = 1$, $e_2 = x$, $e_3 = x^2$, and $f_1 = 1$, $f_2 = x-1$, $f_3 = (x-1)^2$. Let $B_0 = \{e_1, e_2, e_3\}$ and $B_1 = \{f_1, f_2, f_3\}$. We have

$$\begin{aligned} e_1 &= 1 = (1)\cdot f_1 + 0\cdot f_2 + 0\cdot f_3 \quad \Rightarrow \quad (e_1)_{B_2} = [1,0,0] \\ e_2 &= x = 1 + (x-1) = (1)\cdot f_1 + (1)\cdot f_2 + (0)\cdot f_3 \quad \Rightarrow \quad (e_2)_{B_2} = [1,1,0] \\ e_3 &= x^2 = 1 + 2(x-1) + (x-1)^2 = (1)\cdot f_1 + (2)\cdot f_2 + (1)\cdot e_3 \quad \Rightarrow \quad (f_2)_{B_0} = [1,2,1] \end{aligned}$$

The change of basis matrix $C_{B_2B_1}$ is formed by using the coordinate vectors $(f_1)_{B_0}, (f_2)_{B_0}, (f_3)_{B_0}$ as columns.

$$C_{B_1B_2} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

Here is how this matrix is employed. Start with an arbitary coordinate vector w.r.t. to B_0

$$[a_1, a_2, a_3]$$

(which would correspond to the polynomial $a_1 + a_2x + a_3x^2$). Rewrite it as column vector and then multiply from the left by $C_{B_0B_1}$; this should yield the coordinates of the same polynomial w.r.t. to the basis B_1. Indeed

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} a_1 + a_2 + a_3 \\ a_2 + 2a_3 \\ a_3 \end{bmatrix}$$

: So we should have

$$\begin{aligned} a_0 + a_1x + a_2x^2 &= (a_1 + a_2 + a_3) + (a_2 + 2a_3)(x-1) + (a_3)(x-1)^2 \\ &= a_1 + a_2 + a_3 + a_2x + 2a_3x - a_2 - 2a_3 + a_3x^2 - 2a_3x + a_3 \\ &= a_1 + a_2x + a_3x^2 \quad \checkmark \end{aligned}$$

:

3. Use the definition $\det(\mathbf{M}) = \sum_{\sigma \in S_n} \varepsilon(\sigma) M_{1\sigma_1} \cdots M_{2\sigma_2}$ to calculate the determinant of $\mathbf{M} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$

- There are $6 = 3!$ permutations of $[1,2,3]$; namely,

$$[1,2,3], [1,3,2], [2,1,3], [2,3,1], [3,1,2], [3,2,1]$$

We have

$$\begin{aligned} \varepsilon([1,2,3]) &= 1 \quad , \quad \varepsilon([2,3,1]) = 1 \quad , \quad \varepsilon([3,1,2]) = 1 \\ \varepsilon([1,3,2]) &= -1 \quad , \quad \varepsilon([2,1,3]) = -1 \quad , \quad \varepsilon([3,2,1]) = -1 \end{aligned}$$

And so

$$\begin{aligned} \det(\mathbf{M}) &= \varepsilon([1,2,3]) M_{1,1}M_{2,2}M_{3,3} + \varepsilon([1,3,2]) M_{11}M_{23}M_{32} + \varepsilon([2,1,3]) M_{12}M_{21}M_{33} \\ &\quad + \varepsilon([2,3,1]) M_{12}M_{23}M_{31} + \varepsilon([3,1,2]) M_{13}M_{21}M_{32} + \varepsilon([3,2,1]) M_{13}M_{22}M_{31} \\ &= aei - afh - bdi + bfg + cdh - ceg \end{aligned}$$

4. Consider the following matrix

$$\mathbf{M} = \begin{pmatrix} 0 & 1 & 2 & 1 \\ 2 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 \\ 1 & 2 & 1 & 1 \end{pmatrix}$$

(a) Use row reduction to calculate the determinant of $\mathbf{M}$.

$$
\begin{aligned}
\det(\mathbf{M}) &= \det\begin{pmatrix} 0 & 1 & 2 & 1 \\ 2 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 \\ 1 & 2 & 1 & 1 \end{pmatrix} \xrightarrow{R_1 \longleftrightarrow R_4} -\det\begin{pmatrix} 1 & 2 & 1 & 1 \\ 2 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 \end{pmatrix} \\
&\xrightarrow{R_2 \longleftrightarrow R_2 - 2R_1} -\det\begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & -4 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 \end{pmatrix} \xrightarrow{R_2 \longleftrightarrow R_3} +\det\begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & -4 & -1 & 0 \\ 0 & 1 & 2 & 1 \end{pmatrix} \\
&\xrightarrow[R_4 \to R_4 - R_2]{R_3 \to R_3 + 4R_2} +\det\begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 2 & 1 \end{pmatrix} \xrightarrow{R_4 \to R_4 + 2R_3} \det\begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&= (1)(1)(-1)(1) \\
&= -1
\end{aligned}
$$

(b) Use a cofactor expansion to calculate the determinant of $\mathbf{M}$.

- We'll do a cofactor expansion along the third row

$$
\begin{aligned}
\det(\mathbf{M}) &= (-1)^{3+1}(0)\det\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 2 & 1 & 1 \end{pmatrix} + (-1)^{3+2}(1)\det\begin{pmatrix} 0 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix} \\
&\quad + (-1)^{3+3}(0)\det\begin{pmatrix} 0 & 1 & 1 \\ 2 & 0 & 2 \\ 1 & 2 & 1 \end{pmatrix} + (-1)^{3+4}\det\begin{pmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 1 & 2 & 1 \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
&= 0 - \det\begin{pmatrix} 0 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix} + 0 - 0; \\
&= -\left[(-1)^{1+1}(0)\det\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} + (-1)^{1+2}(2)\det\begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix} + (-1)^{1+3}(1)\det\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\right] \\
&= -(0 - 2(2-2) + (1)(2-1)) \\
&= -1
\end{aligned}
$$

5. Determine if the vectors $\mathbf{v}_1 = [0,1,2,1]$, $\mathbf{v}_2 = [1,0,0,2]$, $\mathbf{v}_3 = [2,1,1,1]$ and $\mathbf{v}_4 = [0,0,1,0]$ are linearly independent by calculating a particular determinant.

- The vectors will be linearly independent if and only if the determine of the matrix constructed from their entries is non-zero:

$$
\mathbf{M} = \begin{bmatrix} 0 & 1 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 2 & 0 & 1 & 1 \\ 1 & 2 & 1 & 0 \end{bmatrix}
$$

One finds (e.g. by factor expansion down the fourth column),

$$
\det(\mathbf{M}) = -4 \neq 0
$$

and so the vectors are linearly independent.

6. Consider the matrix

$$\mathbf{M} = \begin{pmatrix} 3 & 0 & 4 \\ -2 & 1 & 1 \\ 3 & 1 & 2 \end{pmatrix}$$

(a) Compute the cofactor matrix $\mathbf{C_M}$ of $\mathbf{M}$.

- We have

$$\begin{aligned} (\mathbf{C_M})_{11} &= (-1)^{1+1} \det \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = 1 \\ (\mathbf{C_M})_{12} &= (-1)^{1+2} \det \begin{pmatrix} -2 & 1 \\ 3 & 2 \end{pmatrix} = 7 \\ (\mathbf{C_M})_{13} &= (-1)^{1+3} \det \begin{pmatrix} -2 & 1 \\ 3 & 1 \end{pmatrix} = -5 \end{aligned}$$

$$\begin{aligned} (\mathbf{C_M})_{21} &= (-1)^{2+1} \det \begin{pmatrix} 0 & 4 \\ 1 & 2 \end{pmatrix} = 4 \\ (\mathbf{C_M})_{22} &= (-1)^{2+2} \det \begin{pmatrix} 3 & 4 \\ 3 & 2 \end{pmatrix} = -6 \\ (\mathbf{C_M})_{23} &= (-1)^{2+3} \det \begin{pmatrix} 3 & 0 \\ 3 & 1 \end{pmatrix} = -3 \end{aligned}$$

$$\begin{aligned} (\mathbf{C_M})_{31} &= (-1)^{3+1} \det \begin{pmatrix} 0 & 4 \\ 1 & 1 \end{pmatrix} = -4 \\ (\mathbf{C_M})_{32} &= (-1)^{3+2} \det \begin{pmatrix} 3 & 4 \\ -2 & 1 \end{pmatrix} = -11 \\ (\mathbf{C_M})_{33} &= (-1)^{3+3} \det \begin{pmatrix} 3 & 0 \\ -2 & 1 \end{pmatrix} = 3 \end{aligned}$$

So

$$\mathbf{C_M} = \begin{pmatrix} 1 & 7 & -5 \\ 4 & -6 & -3 \\ -4 & -11 & 3 \end{pmatrix}$$

(b) Use the result of 7(a) to compute $\mathbf{M}^{-1}$.

- We have

$$\det(\mathbf{A}) = -17$$

and so

$$\mathbf{M}^{-1} = \frac{1}{\det(\mathbf{M})} \mathbf{C}^T = -\frac{1}{17} \begin{pmatrix} 1 & 4 & -4 \\ 7 & -6 & -11 \\ -5 & -3 & 3 \end{pmatrix} = \begin{pmatrix} -\frac{1}{17} & -\frac{4}{17} & \frac{4}{17} \\ -\frac{7}{17} & \frac{6}{17} & \frac{11}{17} \\ \frac{5}{17} & \frac{3}{17} & -\frac{3}{17} \end{pmatrix}$$

7. Solve the following system of linear equations using Crammer's Rule.

$$\begin{aligned} x_1 + 2x_2 - x_3 &= -3 \\ 2x_1 + x_2 + x_3 &= 0 \\ 3x_1 - x_2 + 5x_3 &= 1 \end{aligned}$$

- We have

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 1 & 1 \\ 3 & -1 & 5 \end{pmatrix} \quad , \quad \mathbf{b} = \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$$

and

$$\det(\mathbf{A}) = -3$$

$$\begin{aligned}
\det(\mathbf{B}_1) &= \det\begin{pmatrix} -3 & 2 & -1 \\ 0 & 1 & 1 \\ 1 & -1 & 5 \end{pmatrix} = -15 \\
\det(\mathbf{B}_2) &= \det\begin{pmatrix} 1 & -3 & -1 \\ 2 & 0 & 1 \\ 3 & 1 & 5 \end{pmatrix} = 18 \\
\det(\mathbf{B}_3) &= \det\begin{pmatrix} 1 & 2 & -3 \\ 2 & 1 & 0 \\ 3 & -1 & 1 \end{pmatrix} = 12
\end{aligned}$$

So, Crammer's Rule

$$x_i = \frac{\det(\mathbf{B}_i)}{\det(\mathbf{A})}$$

yields

$$\begin{aligned}
x_1 &= \frac{-15}{-3} = 5 \\
x_2 &= \frac{18}{-3} = -6 \\
x_3 &= \frac{12}{-3} = -4
\end{aligned}$$