MATH 4063-5023
Homework Set 5

1. Suppose that $V$ is a finitely generated vector space and $\phi : V \to W$ is a linear transformation. Show that $im(\phi) \subset W$ is finitely generated.

- Since $V$ is finitely generated it has a finite basis; say, $B = \{v_1, \ldots, v_n\}$ is a basis for $V$. Now

$$\begin{aligned} im(\phi) &\equiv \{w \in W \mid w = \phi(v) \quad , \quad v \in V\} \\ &= \{w \in W \mid w = \phi(\alpha_1 v_1 + \cdots + \alpha_n v_n) \qquad , \qquad \alpha_i \in \mathbb{F}\} \end{aligned}$$

since each vector in $V$ can be expressed as a linear combination of the $v_i \in B$. But then

$$\begin{aligned} im(\phi) &= \{w \in W \mid w = \alpha_1 \phi(v_1) + \cdots + \alpha_n \phi(v_n) \quad , \quad \alpha_i \in \mathbb{F}\} \qquad \text{since } \phi \text{ is a linear transformation} \\ &= span(\phi(v_1), \ldots, \phi(v_n)) \end{aligned}$$

So $im(\phi)$ is generated by $\phi(v_1), \ldots, \phi(v_n)$. □

2. Suppose $S$ is a subspace of a finitely generated vector space $V$. Show that $V/S$ is finitely generated.

- Let $p_S : V \to V/S$ be the canonical projection. This is a linear transformation between $V$ and $V/S$ with image $V/S$. Therefore, by Problem 1 above, since $V$ is finitely generated, the image $V/S$ of $p_S$ must be finitely generated. □

3. Suppose $S$ is a subspace of a finitely generated vector space $V$, find a basis for $V/S$.

- Since $S$ is a subspace of a f.g. vector space, $S$ has a basis $B_S = \{b_1, \ldots, b_k\}$. Moreover, the basis for the subspace $S$ can always be extended to a basis for $V$. Let

$$B_V = \left\{b_1, \ldots, b_k, b'_{k+1}, \ldots, b'_n\right\}$$

be such a basis for $V$. From Problems (1) and (2) above, we know that

$$V/S = im(p_S) = span\left(p_S(b_1), \ldots, p_S(b_k), p_S\left(b'_{k+1}\right), \ldots, p_S(b'_n)\right)$$

Now the for $1 \leq i \leq k$, $b_i \in S = \ker(p_S)$ and so

$$V/S = span\left(\mathbf{0}_{V/S}, \ldots, \mathbf{0}_{V/S}, p_S\left(b'_{k+1}\right), \ldots, p_S(b'_n)\right) = span\left(p_S\left(b'_{k+1}\right), \ldots, p_S(b'_n)\right)$$

So the vectors $p_S\left(b'_{k+1}\right), \ldots, p_S(b'_n)$ span $V/S$. I clain they are also linearly independent. Indeed,

$$\begin{aligned} \mathbf{0}_{V/S} &= \alpha_{k+1} p_S\left(b'_{k+1}\right) + \cdots + \alpha_n p_S(b'_n) \\ &= p_S\left(\alpha_{k+1} b'_{k+1} + \cdots + \alpha_n b'_n\right) \\ &\Rightarrow \qquad \alpha_{k+1} b'_{k+1} + \cdots \alpha_n b'_n \in \ker(p_S) = S \end{aligned}$$

But a vector $a_1 b_1 + \cdots + a_k b_k + \alpha_{k+1} b'_{k+1} + \cdots + a_n b'_n$ can be in $S$ only if $0 = a_{k+1},\ 0 = a_{k+2},\ \ldots, 0 = a_n$. Hence,

$$\mathbf{0}_{V/S} = \alpha_{k+1} p_S\left(b'_{k+1}\right) + \cdots + \alpha_n p_S(b'_n) \quad \Rightarrow \quad 0 = \alpha_{k+1} = \alpha_{k+2} = \cdots = \alpha_n$$

hence the vectors $p_S\left(b'_{k+1}\right), \ldots, p_S(b'_n)$ are linearly independent. □

4. Suppose $S$ is a subspace of a finitely generated vector space $V$, show that $\dim(V) = \dim(S) + \dim(V/S)$.

- From the theory of linear transformations we have, since the canonical projection $p_S : V \to V/S$ is a linear transformation

$$\dim(V) = \dim(im(p_S)) + \dim(\ker(p_S))$$

On the other hand,

$$\begin{aligned} \ker(p_S) &= S \\ im\,(p_S) &= V/S \end{aligned}$$

and so

$$\dim(V) = \dim(V/S) + \dim S \quad .$$

$\square$

5. Suppose $\phi : V \to W$ is a linear transformation between two finite-dimensional vector spaces. Show that $im\,(\phi)$ is isomorphic to $V/\ker(\phi)$. (Hint: two finite-dimensional vector spaces are isomorphic if and only if they have the same dimension.)

- Regarding $\ker\phi$ as a subspace of $V$ we have a canonical projection $p_{\ker\phi} : V \to V/\ker\phi$, and from Problem 4 above we have

(i) $$\dim V = \dim(V/\ker\phi) + \dim(\ker\phi)$$

On the other hand, from our general theory of linear transformations

(ii) $$\dim(V) = \dim(im\,(\phi)) + \dim(\ker\phi)$$

Comparing (i) with (ii) we conclude

$$\dim(V/\ker\phi) = \dim(im\,(\phi))$$

Because $V/\ker\phi$ and $im\,(\phi)$ are finite-dimensional and share the same dimension, they must be isomorphic. $\square$