MATH 4063-5023

Solutions to Homework Set 3

1. Test for the solvability of the following linear systems (over $\mathbb{R}$). If the system is solvable, then express the general solution in the form of $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_0$ where $\mathbf{x}_0$ is a particular solution of the given linear system and $\mathbf{x}_0$ is the general solution of the corresponding homogeneous linear system (see the tail end of Lecture 6).

(a)

$$\begin{aligned} x_1 + x_2 + x_3 &= 8 \\ x_1 + x_2 + x_4 &= 1 \\ x_1 + x_3 + x_4 &= 14 \\ x_2 + x_3 + x_4 &= 14 \end{aligned}$$

- Reducing the augmented matrix for this system to reduced row echelon form we get

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 8 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 14 \\ 0 & 1 & 1 & 1 & 14 \end{array}\right] \quad \rightarrow \quad \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -\frac{5}{3} \\ 0 & 1 & 0 & 0 & -\frac{5}{3} \\ 0 & 0 & 1 & 0 & \frac{34}{3} \\ 0 & 0 & 0 & 1 & \frac{13}{3} \end{array}\right]$$

From the augmented matrix in row echelon, we can read off the unique solution

$$\begin{aligned} x_1 &= -\frac{5}{3} \\ x_2 &= -\frac{2}{3} \\ x_3 &= \frac{34}{3} \\ x_4 &= \frac{13}{3} \end{aligned}$$

So

$$\mathbf{x} = \left[-\frac{5}{3}, -\frac{2}{3}, \frac{34}{3}, \frac{13}{3}\right]$$

is the unique solution vector.

(b)

$$\begin{aligned} 2x_1 + x_2 + 3x_3 - x_4 &= 1 \\ 3x_1 + x_2 - 2x_3 + x_4 &= 0 \\ 2x_1 + x_2 - x_3 + 2x_4 &= -1 \end{aligned}$$

- Applying our augmented matrix method we get

$$\left[\begin{array}{cccc|c} 2 & 1 & 3 & -1 & 1 \\ 3 & 1 & -2 & 1 & 0 \\ 2 & 1 & -1 & 2 & -1 \end{array}\right] \quad \rightarrow \quad : \left[\begin{array}{cccc|c} 1 & 0 & 0 & -\frac{7}{4} & \frac{3}{2} \\ 0 & 1 & 0 & \frac{19}{4} & -\frac{7}{2} \\ 0 & 0 & 1 & -\frac{3}{4} & \frac{1}{2} \end{array}\right]$$

which leads to the following equations for the solution

$$\begin{aligned} x_1 - \frac{7}{4}x_4 &= \frac{3}{2} \\ x_2 + \frac{19}{4}x_4 &= -\frac{7}{2} \\ x_3 - \frac{3}{4}x_4 &= \frac{1}{2} \end{aligned}$$

Following our convention to regard variables as free parameters in the solution whenever they correspond to a column in the reduced row echelon form of the augmented matrix that does not contain a pivot, we take x_4 to be a free parameter, say $x_4 = s$. The above equations then allow us to also express x_1, x_2 and x_3 in terms of s

$$\begin{aligned} x_1 &= \frac{3}{2} + \frac{7}{4}s \\ x_2 &= -\frac{7}{2} - \frac{19}{4}s \\ x_3 &= \frac{1}{2} + \frac{3}{4}s \\ x_4 &= s \end{aligned}$$

So

$$\begin{aligned} \mathbf{x} &= \left[\frac{3}{2} + \frac{7}{4}s, -\frac{7}{2} - \frac{19}{4}s, \frac{1}{2} + \frac{3}{4}s, s\right] \\ &= \left[\frac{3}{2}, -\frac{7}{2}, \frac{1}{2}, 0\right] + s\left[\frac{7}{4}, -\frac{19}{4}, \frac{3}{4}, 1\right] \end{aligned}$$

Here $\left[\frac{3}{2}, -\frac{7}{2}, \frac{1}{2}, 0\right]$ is a particular solution of the original system and $s\left[\frac{7}{4}, -\frac{19}{4}, \frac{3}{4}, 1\right]$ is a solution of the corresponding homogeneous system.

(c)

$$\begin{aligned} x_1 + 4x_2 + 3x_3 &= 1 \\ 3x_1 + x_3 &= 1 \\ 4x_1 + x_2 + 2x_3 &= 1 \end{aligned}$$

- Following again the augmented matrix method

$$[\mathbf{A}|\mathbf{b}] = \left[\begin{array}{ccc|c} 1 & 4 & 3 & 1 \\ 3 & 0 & 1 & 1 \\ 4 & 1 & 2 & 1 \end{array}\right] \quad \rightarrow \quad \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{3} & 0 \\ 0 & 1 & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] = [\mathbf{A}'|\mathbf{b}']$$

Noting the final column vector $\mathbf{b}'$ lies outside the column space of the final coefficient matrix $\mathbf{A}'$, we can conclude that the system has no solution (See Theorem 7.2.)

(d)

$$\begin{aligned} -x_1 + 2x_2 + x_3 + 4x_4 &= 0 \\ 2x_1 + x_2 - x_3 + x_4 &= 1 \end{aligned}$$

- Once again, row reducing the augmented matrix for the system we find

$$\left[\begin{array}{cccc|c} -1 & 2 & 1 & 4 & 0 \\ 2 & 1 & -1 & 1 & 1 \end{array}\right] \quad \rightarrow \quad \left[\begin{array}{cccc|c} 1 & 0 & -\frac{3}{5} & -\frac{2}{5} & \frac{2}{5} \\ 0 & 1 & \frac{1}{5} & \frac{9}{5} & \frac{1}{5} \end{array}\right]$$

From the reduced row echelon form of the augmented matrix we see that since columns 3 and 4 lack pivots, we should regard x_3 and x_4 as being free parameters in the solution.

Setting $x_3 = s_1$ and $x_4 = s_2$, and using the equations of the R.R.E.F. of the augmented matrix

$$\left.\begin{array}{l} x_1 - \frac{2}{5}x_3 - \frac{3}{5}x_4 = \frac{2}{5} \\ x_2 + \frac{1}{5}x_3 + \frac{9}{5}x_4 = \frac{1}{5} \end{array}\right\} \quad \Rightarrow \quad \left\{\begin{array}{l} x_1 = \frac{2}{5} + \frac{3}{5}s_1 + \frac{2}{5}s_2 \\ x_2 = \frac{1}{5} - \frac{1}{5}s_1 - \frac{9}{5}s_2 \end{array}\right.$$

We can write down a generic solution vector as

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{2}{5} + \frac{3}{5}s_1 + \frac{2}{5}s_2 \\ \frac{1}{5} - \frac{1}{5}s_1 - \frac{9}{5}s_2 \\ s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} \frac{2}{5} \\ \frac{1}{3} \\ 0 \\ 0 \end{bmatrix} + s_1 \begin{bmatrix} \frac{3}{5} \\ -\frac{1}{5} \\ 1 \\ 0 \end{bmatrix} + s_2 \begin{bmatrix} \frac{2}{5} \\ -\frac{9}{5} \\ 0 \\ 1 \end{bmatrix}$$

The last expression presents the solution in the form of a particular solution $\mathbf{x}_p = \left[\frac{2}{5}, \frac{1}{5}, 0, 0\right]$ plus a solution of the corresponding homogeneous system (I note that the vectors $\left[\frac{3}{5}, -\frac{1}{5}, 1, 0\right]$ and $\left[\frac{2}{5}, -\frac{9}{5}, 0, 1\right]$ actually constitute a basis for the solution space of the corresponding homogeneous system.)

2. Prove that an $n \times m$ system of homogeneous equations has a non-trivial solution if and only if the rank of the coefficient matrix is less than m.

- By Corollary 9.4, the solution space of an $n \times m$ system is equal to the number of columns m of the coefficient matrix minus its rank. This implies in particular that
$$m - r \geq 0.$$
Now if $m = r$, the solution space is 0-dimensional. But there is only one 0-dimensional subspace of $\mathbb{F}^m$, namely $\{\mathbf{0}\}$. On the other hand, it also clear that in order to have a non-trivial solution (a solution other than $\mathbf{x} = \mathbf{0}$), we will need a solution space with positive dimension. Thus, for non-trivial solutions we requre $r < m$.

3. Find a set of homogeneous linear equations whose solution set is the subspace of $\mathbb{R}^3$ generated by the vectors $[2, 1, -3], [1, -1, 0]$ and $[1, 3, -4]$.

- We follow the method of Example 10.7. Let S^* be the solution set of (the equations corresponding to the condition that each $\mathbf{x}$ in S is perpendicular to the generators of $S = span\left([2, 1, -3], [1, -1, 0], [1, 3, -4]\right)$>
$$\begin{aligned} 2x_1 + x_2 - 3x_3 &= 0 \\ x_1 - x_2 &= 0 \\ x_1 + 3x_2 - 4x_3 &= 0 \end{aligned}$$
A basis for the solution set can be obtained via our row reduction method
$$\left[\begin{array}{ccc|c} 2 & 1 & -3 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & 3 & -4 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \left\{\begin{array}{l} x_1 = x_3 \\ x_2 = x_3 \end{array}\right.$$
$$\Rightarrow \quad \mathbf{x} \in span\left([1, 1, 1]\right)$$
To find the equations for S we demand that each $\mathbf{y} \in S$ be perpendicular to such an $\mathbf{x} \in S^*$. Thus,
$$S = S^{**} = \text{solution set of } x_1 + x_2 + x_3 = 0$$

4. Let S be a subspace of a vector space V. Prove that if $\mathbf{p}$ and $\mathbf{q}$ are vectors belonging to the hyperplane $M = H_{\mathbf{x}_0, S} = \{\mathbf{x}_0 + \mathbf{s} \mid \mathbf{s} \in S\}$, then the line through $\mathbf{p}$ and $\mathbf{q}$ lies entirely in M.

- Since $\mathbf{p}, \mathbf{q} \in M$, we have
$$\begin{aligned} \mathbf{p} &= \mathbf{x}_0 + \mathbf{s} \qquad \text{for some } \mathbf{s} \in S \\ \mathbf{q} &= \mathbf{x}_0 + \mathbf{s}' \qquad \text{for some } \mathbf{s}' \in S \end{aligned}$$
The line through $\mathbf{p}$ and $\mathbf{q}$ can be formed by starting at $\mathbf{p}$ and then heading off in the direction $\mathbf{q} - \mathbf{p}$.
$$\text{line through } \mathbf{p} \text{ and } \mathbf{q} = \{\mathbf{x} = \mathbf{p} + t\left(\mathbf{q} - \mathbf{p}\right) \mid t \in \mathbb{R}\}$$
Note that when $t = 0$, $\mathbf{x} = \mathbf{p}$ and that when $t = 1$ we have $\mathbf{x} = \mathbf{q}$, so this is certainly the line we want. Now let
$$\mathbf{y} = \mathbf{p} + i\left(\mathbf{q} - \mathbf{p}\right)$$
be an arbitary point on this line. We then have
$$\begin{aligned} \mathbf{y} &= (\mathbf{x}_0 + \mathbf{s}) + t\left((\mathbf{x}_0 + \mathbf{s}') - (\mathbf{x}_0 + \mathbf{s})\right) \\ &= \mathbf{x}_0 + \mathbf{s} + t\mathbf{x}_0 + t\mathbf{s}' - t\mathbf{x}_0 - t\mathbf{s} \\ &= \mathbf{x}_0 + (1 - t)\,\mathbf{s} + t\mathbf{s}' \end{aligned}$$

which is in M since $(1-t)\,\mathbf{s}+t\mathbf{s}'\in S$ if $\mathbf{s},\mathbf{s}'\in S$.

5. Find the point in $\mathbb{R}^3$ where the line joining the points $[1,-1,0]$ and $[-2,1,1]$ pierces the plane $3x_1 - x_2 + x_3 = 1$.

- The points on the line joining $[1,-1,0]$ and $[-2,1,1]$ are given by
$$[x_1,x_2,x_3] = [1,-1,0] + t\,([-2,1,1]-[1,-1,0]) = [1-3t,-1+2t,t]$$
Or
$$\mathbf{x} = \left[\begin{array}{c} 1-3t \\ -1+2t \\ t \end{array}\right]$$
Such a vector would also arise as the general solution to a linear system with augmented matrix in reduced row echelon form
$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 1 \\ 0 & 1 & -2 & -1 \end{array}\right]$$
and so the linear system
$$\begin{aligned} x_1 + 3x_3 &= 1 \\ x_2 - 2x_3 &= -1 \end{aligned}$$
To find the intersection of the given line with the plane $3x_1 - x_2 + x_3 = 1$ we simply try to solve all three equations simultaneously
$$\begin{aligned} x_1 + 3x_3 &= 1 \\ x_2 - 2x_3 &= -1 \\ 3x_1 - x_2 + x_3 &= 1 \end{aligned}$$
Following our by now standard procedure
$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 1 \\ 0 & 1 & -2 & -1 \\ 3 & -1 & 1 & 1 \end{array}\right] \qquad \rightarrow \qquad \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{10} \\ 0 & 1 & 0 & -\frac{2}{5} \\ 0 & 0 & 1 & \frac{3}{10} \end{array}\right]$$
we can conclude that unique solution to our problem is the vector with coordinates
$$x_1 = \frac{1}{10} \quad , \quad x_2 = -\frac{2}{5} \quad , \quad x_3 = \frac{1}{10}$$