MATH 4063-5023
Solutions to Homework Set 2

1. Use elementary row operations to systematically transform the following matrices to row echelon form.

(a) $\left(\begin{array}{cccc} 1 & 2 & 1 & 0 \\ 2 & 3 & 0 & -1 \\ 1 & 2 & 1 & -1 \end{array}\right)$

- We have

$$\left(\begin{array}{cccc} 1 & 2 & 1 & 0 \\ 2 & 3 & 0 & -1 \\ 1 & 2 & 1 & -1 \end{array}\right) \quad \begin{array}{c} R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 - R_1 \end{array} \quad \rightsquigarrow \quad \left(\begin{array}{cccc} 1 & 2 & 1 & 0 \\ 0 & -1 & -2 & -1 \\ 0 & 0 & 0 & -1 \end{array}\right)$$

The matrix on the far right is in row echelon form

(b) $\left(\begin{array}{ccc} 1 & 1 & 2 \\ 0 & 2 & -1 \\ 1 & 2 & 4 \end{array}\right)$

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$$\begin{array}{cccccc} \left(\begin{array}{ccc} 1 & 1 & 2 \\ 0 & 2 & -1 \\ 1 & 2 & 4 \end{array}\right) & R_3 & \rightarrow & R_3 - R_1 & \rightsquigarrow & \left(\begin{array}{ccc} 1 & 1 & 2 \\ 0 & 2 & -1 \\ 0 & 1 & 2 \end{array}\right) \\ & R_1 & \longleftrightarrow & R_2 & \rightsquigarrow & \left(\begin{array}{ccc} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 2 & -1 \end{array}\right) \\ & R_3 & \rightarrow & 2R_2 & \rightsquigarrow & \left(\begin{array}{ccc} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & -3 \end{array}\right) \end{array}$$

The last matrix is in row echelon form.

(c) $\left(\begin{array}{cccc} 2 & -1 & 1 & 2 \\ 3 & 0 & 1 & 1 \\ 0 & 3 & -1 & -4 \end{array}\right)$

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$$\begin{array}{cccccc} \left(\begin{array}{cccc} 2 & -1 & 1 & 2 \\ 3 & 0 & 1 & 1 \\ 0 & 3 & -1 & -4 \end{array}\right) & R_2 & \rightarrow & R_2 - \dfrac{2}{3}R_3 & \rightsquigarrow & \left(\begin{array}{cccc} 2 & -1 & 1 & 2 \\ 0 & \frac{2}{3} & \frac{1}{3} & -\frac{1}{3} \\ 0 & 3 & -1 & -4 \end{array}\right) \\ & R_3 & \rightarrow & R_3 - \dfrac{9}{2}R_2 & \rightsquigarrow & \left(\begin{array}{cccc} 2 & -1 & 1 & 2 \\ 0 & \frac{2}{3} & \frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & -\frac{5}{2} & -\frac{5}{2} \end{array}\right) \end{array}$$

The last matrix is in row echelon form.

2. Test the following sets of vectors for linear independence.

(a) $\{[-1,1],[1,2],[1,3]\}$

- Note that each vector lies in the span of $\{[1,0],[0,1]\}$. But any set of three vectors in a subspace generated by two vectors must be linearly independent (see Theorem 3.3 in the online lecture notes). Thus, three stated vectors must be linearly dependent.

(b) $\{[0,1,1,2],[3,1,5,2],[-2,1,0,1],[1,0,3,-1]\}$

- We write the four given vectors as the rows of a 4×4 matrix.

$$\begin{pmatrix} 0 & 1 & 1 & 2 \\ 3 & 1 & 5 & 2 \\ -2 & 1 & 0 & 1 \\ 1 & 0 & 3 & -1 \end{pmatrix}$$

This matrix row reduces to

$$:\begin{pmatrix} 5 & 0 & 0 & 4 \\ 0 & 5 & 0 & 13 \\ 0 & 0 & 5 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and so the subspace generated by the given four vectors is coincides with the subspace generated by the three non-zero rows of the matrix in row echelon form. Therefore, by Theorem 3.3, the original set of four vectors must be linearly dependent.

(c) $\{[1,1,0,0,1],[-1,1,1,0,0],[2,1,0,1,1],[0,-1,-1,-1,0]\}$

- We write the four given vectors as the rows a 4×5 matrix

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ -1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 1 & 1 \\ 0 & -1 & -1 & -1 & 0 \end{pmatrix}$$

This matrix can be row reduced to

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The zero row at the bottom indicates the original set of vectors was linearly independent (essentially the same argument as in part (b)).

3. Test the following sets of polynomials for linear dependence.

(a) $\left\{x^2+2x+1, 2x+1, 2x^2-2x-1\right\}$

- Each of polynomials lies in the vector space $\mathcal{P}_2$ of polynomials of degree ≤ 2. If we adopt the natural basis $\left\{1, x, x^2\right\}$ for $\mathcal{P}_2$, the corresponding coefficient matrix is

$$\begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 0 \\ -1 & -2 & 2 \end{pmatrix}$$

This matrix row reduces to

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

Since the row echelon form of the coefficient matrix has a zero row, it must be that the original set of 3 polynomials is a linearly dependent set. (See Theorem 4.15 (c).)

(b) $\left\{1, x-1, (x-1)^2, (x-1)^3\right\}$

- Each polynomial in the list lies in $\mathcal{P}_3 = span\left\{1, x, x^2, x^3\right\}$, which is a linearly independent set of four polynomials. The coefficient vectors of these polynomials with respect to the basis $\left\{x^3, x^2, x, 1\right\}$ of $\mathcal{P}_3$ are

$$\begin{aligned} 1 &= 1 \cdot 1 \quad \Rightarrow \quad [0,0,0,1] \\ x-1 &= (1)\cdot x + (-1)\cdot 1 \quad \Rightarrow \quad [0,0,1,-1] \\ (x-1)^2 &= (1)\cdot x^3 + (-2)\cdot x + (1)\cdot 1 \quad \Rightarrow \quad [0,1,-2,1] \\ (x-1)^3 &= (1)\cdot x^3 + (-3)\cdot x^2\,(1)\cdot 1 + (3)\cdot x + (-1)\cdot 1 \quad \Rightarrow \quad [-1,3,-3,1] \end{aligned}$$

So the coefficient matrix of these four polynomials with respect to the basis $\left\{x^3, x^2, x, 1\right\}$ of $\mathcal{P}_3$ is

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & -2 & 1 \\ 1 & -3 & 3 & -1 \end{pmatrix}$$

which is row equivalent to

$$\begin{pmatrix} 1 & -3 & 3 & -1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

which is in row echelon form. By Theorem 4.15(c) we can conclude that the original set of four polynomials is a linearly independent set.

4. Determine if $[1,1,1]$ belongs to the subspace of $\mathbb{R}^3$ generated by $[1,3,4], [4,0,1], [3,1,2]$. Explain your reasoning.

- Let's first consider the span of the three vectors $\in \mathbb{R}^3$. Let's first find a basis for $span_{\mathbb{R}}\left([1,3,4], [4,0,1], [3,1,2]\right)$. This we do by writing the vectors as the rows of a 3×3 matrix and row reducing that matrix to row echelon form:

$$\begin{pmatrix} 1 & 3 & 4 \\ 4 & 0 & 1 \\ 3 & 1 & 2 \end{pmatrix} \quad \text{row reduces to} \quad \begin{pmatrix} 1 & 0 & \frac{1}{4} \\ 0 & 1 & \frac{5}{4} \\ 0 & 0 & 0 \end{pmatrix}$$

From this we conclude that any vector in $span_{\mathbb{R}}\left([1,3,4], [4,0,1], [3,1,2]\right)$ can be expressed as a vector in $span_{\mathbb{R}}\left(\left[1,0,\frac{1}{4}\right], \left[0,1,\frac{5}{4}\right]\right)$. We now check to see if $[1,1,1]$ when adjoined to $\left\{\left[1,0,\frac{1}{4}\right], \left[0,1,\frac{5}{4}\right]\right\}$ forms a linearly dependent set. Thus we look at the matrix

$$\begin{pmatrix} 1 & 0 & \frac{1}{4} \\ 0 & 1 & \frac{5}{4} \\ 1 & 1 & 1 \end{pmatrix}$$

, row echelon form: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ which row reduces to

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

i.e. a matrix with three linearly independent row vectors. We conclude that $[1,1,1]$ lies outside $span_{\mathbb{R}}\left(\left[1,0,\frac{1}{4}\right], \left[0,1,\frac{5}{4}\right]\right) = span_{\mathbb{R}}\left([1,3,4], [4,0,1], [3,1,2]\right)$. Thus, $[1,1,1]$ can not be written as a linear combination of $[1,3,4]$, $[4,0,1]$, and $[3,1,2]$.

5. Prove every subspace S of a finitely generated vector space is finitely generated and that $\dim S \leq \dim T$ with equality only if $S = T$.

- Let S be a subspace of a finitely generated vector space V. Since V is finitely generated, V has a basis $\{b_1, \ldots, b_n\}$ (see Theorem 5.2). Pick a nonzero element s_1 of S. If $S = span\,(s_1)$ then we are done as s_1 generates S. Otherwise, there must be another vector $s_2 \in S$ such that $s_2 \notin span\,(s_1)$. If $S = span\,(s_1, s_2)$, then S is generated by s_1 and s_2 and we are done. Note that since $s_2 \notin span\,(s_1)$, s_1 and s_2 are linearly indepenent. If $S \neq span\,(s_1, s_2)$, there must be a third linearly independent vector s_3 and either $S = span\,(s_1, s_2, s_3)$ or there is a vector $s_4 \in S$ that lies outside of $span\,(s_1, s_2, s_3)$ and so is linearly independent of s_1, s_2, s_3.

  In this fashion we either end up with a finite list of linearly independent generators for S (we stop the process of finding a new s_i once $S = span\,(s_1, \ldots, s_{i-1})$), or we have an infinite sequence $\{s_1, s_2, \ldots, \}$ of linearly independent elements of S. But the latter situation is impossible - because by Theorem 3.3 we can not have more than $n = \dim V$ linearly independent vectors in V. Thus, the process described in the first paragraph must terminate after finitely many steps; say after m steps, with $m \leq n$.

  Now suppose the algorithm terminates after exactly n steps and suppose $V \neq span\,(s_1, \ldots, s_n)$. Then there must be a non-zero element $v \in V$ that lies outside the span of $s_1, \ldots, s_n$. On the other hand, by Theorem 3.3, the vectors $\{s_1, \ldots, s_n, v\}$ being a set of $n+1$ vectors in a vector space generated by n vectors must be a linearly dependent set. We can then in the situation of Lemma 5.1, which tells us that in fact, v must be expressible as a linear combination of $s_1, \ldots, s_n$, which contradicts our hypothesis that $V \neq span\,(s_1, \ldots, s_n)$. We thus conclude that whenever a subspace S has the same number of linearly independent generators as the vector space V that contains it, we must have $S = V$.

6. Let $\mathbb{F}$ be a field with exactly two elements (it will be isomorphic to $\mathbb{Z}_2$) and let V be a 2-dimensional vector space over $\mathbb{F}$. How many vectors are there in V? How many different bases are there for V?

- The main point of this problem is to do as much as possible in the abstract, without using any explicit realization of $\mathbb{F}$ or V. Thus, even though one might know that any field with only two elements is isomorphic to $\mathbb{Z}_2$, we will avoid doing calculations in $\mathbb{Z}_2$.

  Instead, we'll use the field axioms to deduce that $\mathbb{F} = \{0_{\mathbb{F}}, 1_{\mathbb{F}}\}$. For as a field $\mathbb{F}$ has to contain an additive identity $0_{\mathbb{F}}$ and a multiplicative identity $1_{\mathbb{F}}$, It is easy to see that these can be the same if and only if $\mathbb{F}$ is a 1-element field. Therefore, our 2-element field has to be precisely $\{0_{\mathbb{F}}, 1_{\mathbb{F}}\}$.

  Next, let's consider a general 2-dimensional vector space V over $\mathbb{F}$. Since it is 2-dimensional, it has a basis (otherwise it would not even have a dimension). Let $B = \{\mathbf{b}_1, \mathbf{b}_2\}$ be such a basis. The problem now is to count the other possible bases.

  Since B is a basis for V we have

$$\begin{aligned} V &= span_{\mathbb{F}}\,(\mathbf{b}_1, \mathbf{b}_2) \\ &= \{a_1\mathbf{b}_1 + a_2\mathbf{b}_2 \mid a_1, a_2 \in \mathbb{F}\} \end{aligned}$$

  Since $\mathbb{F}$ is finite, we can list all possible elements of V :

$$\begin{aligned} V &= \{0_{\mathbb{F}} \cdot \mathbf{b}_1 + 0_{\mathbb{F}}\mathbf{b}_2, 1_{\mathbb{F}} \cdot \mathbf{b}_1 + 0_{\mathbb{F}} \cdot \mathbf{b}_2, 0_{\mathbb{F}} \cdot \mathbf{b}_1 + 1_{\mathbb{F}} \cdot \mathbf{b}, 1_{\mathbb{F}} \cdot \mathbf{b}_1 + 1_{\mathbb{F}} \cdot \mathbf{b}\} \\ &= \{\mathbf{0}_V, \mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_1 + \mathbf{b}_2\} \end{aligned}$$

  So we have exactly 4 vectors in V. Since V is 2-dimensional, any basis for V will consist of precisely 2 vectors. We can exclude right away the pairs $\{\mathbf{u}, \mathbf{v}\}$ where one of the vectors is $\mathbf{0}_V$. Because it is trivial to construct a dependence relation involving the 0-vector. Thus, we need to look at the other possible pairs

$$\{\mathbf{b}_1, \mathbf{b}_2\} \quad , \quad \{\mathbf{b}_1, \mathbf{b}_1 + \mathbf{b}_2\} \quad , \quad \{\mathbf{b}_2, \mathbf{b}_1 + \mathbf{b}_2\}$$

The first pair is already known to be a basis. To see that the second pair is a linearly independent set, suppose

$$\begin{aligned}
\alpha\mathbf{b}_1+\beta\left(\mathbf{b}_1+\mathbf{b}_2\right) &= \mathbf{0}_V \\
&\Rightarrow \quad (\alpha+\beta)\,\mathbf{b}_1+\beta\mathbf{b}_2=\mathbf{0} \\
&\Rightarrow \quad \alpha+\beta=0_{\mathbb{F}} \qquad \text{and} \quad \beta=0_{\mathbb{F}} \qquad \text{since } \{\mathbf{b}_1,\mathbf{b}_2\} \text{ is a basis} \\
&\Rightarrow \quad \alpha=0_{\mathbb{F}} \text{ and } \beta=0_{\mathbb{F}} \\
&\Rightarrow \quad \{\mathbf{b}_1,\mathbf{b}_1+\mathbf{b}_2\} \text{ are linearly independent} \\
&\Rightarrow \quad \{\mathbf{b}_1,\mathbf{b}_1+\mathbf{b}_2\} \text{ is a basis for } V
\end{aligned}$$

Similarly, one shows that last pair $\{\mathbf{b}_2,\mathbf{b}_1+\mathbf{b}_2\}$ is a set of linearly independent vectors and so much be a basis for the two dimensional space V.

We thus find that there are exactly 3 bases for V.