Math 4063-5023
SOLUTIONS TO SECOND EXAM
November 18, 2014

1. Definitions. Write down the precise definitions of the following notions. (3 pts each)

(a) **vector space homomorphism** (a.k.a. linear transformation)

- A vector space homomorphism is a mapping $\phi : V \to W$ between two vector spaces such that

$$\begin{aligned} \phi(\lambda v) &= \lambda \phi(v) \qquad \forall\, \lambda \in \mathbb{F} \text{ and } \forall\, v \in V && \text{(i)} \\ \phi(v+u) &= \phi(v) + \phi(u) \qquad \forall\, v, u \in V && \text{(ii)} \end{aligned}$$

(b) **kernel of a vector space homomorphism**

- The kernal of a vector space homomorphism $\phi : V \to W$ is the set

$$\ker(\phi) = \{ v \in V \mid \phi(v) = \mathbf{0}_W \}$$

(c) **image of a vector space homomorphism**

- The image of a vector space homomorphism $\phi : V \to W$ is the set

$$im(\phi) = \{ w \in W \mid w = \phi(v) \quad \text{for some } v \in V \}$$

(d) **a surjective vector space homorphism**

- A vector space homomorphism $\phi : V \to W$ is surjective if $im(\phi) = W$.

(e) **an injective vector space homomorphism**

- A vector space homomorphism $\phi : V \to W$ is injective if

$$\phi(v) = \phi(u) \quad \Rightarrow \quad v = u \quad .$$

2. Consider the linear transformation $\phi : \mathbb{R}^3 \to \mathbb{R}^2$ given by

$$\phi([a_1, a_2, a_2]) = [a_1 + a_2 + a_3, a_1 - a_3]$$

(a) (5 pts) Find the matrix representing ϕ (using the standard bases for $\mathbb{R}^3$ and $\mathbb{R}^2$)

$$\mathbf{A}_\phi = \left[\begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \phi(\mathbf{e}_1) & \phi(\mathbf{e}_2) & \phi(\mathbf{e}_3) \\ \downarrow & \downarrow & \downarrow \end{array} \right] = \left[\begin{array}{ccc} 1 & 1 & 1 \\ 1 & 0 & -1 \end{array} \right]$$

(b) (5 pts) What is the kernel of ϕ?

- – $\ker(\phi)$ will be the solution set of $\mathbf{A}_\phi \mathbf{x} = \mathbf{0}$. The reduced row echelon form of $\mathbf{A}_\phi$ is easily seen to be $\left[\begin{array}{ccc} 1 & 0 & -1 \\ 0 & 1 & 2 \end{array} \right]$, and so the solution set of $\mathbf{A}_\phi \mathbf{x} = \mathbf{0}$ will coincide with the solutions of

$$\left. \begin{array}{r} x_1 - x_3 = 0 \\ x_2 + 2x_3 = 0 \end{array} \right\} \quad \Rightarrow \quad \left\{ \begin{array}{c} x_1 = x_3 \\ x_2 = -2x_3 \end{array} \right. \quad \Rightarrow \quad \mathbf{x} = \left[\begin{array}{c} x_3 \\ -2x_3 \\ x_3 \end{array} \right] = x_3 \left[\begin{array}{c} 1 \\ -2 \\ 1 \end{array} \right]$$

So

$$\ker(\phi) = span([1, -2, 1])$$

(c) (5 pts) What is the image of ϕ?

- $im\left(\phi\right)$ will be the column space of $\mathbf{A}_{\phi}$. It fairly obvious that any pair of the columns of $\mathbf{A}_{\phi}$ will be a set of linearly independent generators for $ColSp\left(\mathbf{A}_{\phi}\right)$. And so, for example,

$$im\left(\phi\right) = s\not p an\left(\left[1,0\right],\left[0,1\right]\right)$$

3. Let P_2 be the vector space of polynomials in x of degree ≤ 2. Set

$$T(p) = x\frac{d^2p}{dx^2}$$

(a) (5 pts) Show that $T : P_2 \to P_2$ is a linear transformation.

- It suffices to show $T(\alpha p_1 + \beta p_2) = \alpha T(p_1) + \beta T(p_2)$ for any two polynomials $p_1, p_2 \in P_2$ and any pair of constants α, β. We have
$$T(\alpha p_1 + \beta p_2) = x\frac{d^2}{dx^2}(\alpha p_1 + \beta p_2) = x\alpha\frac{d^2p_1}{dx^2} + x\beta\frac{d^2p_2}{dx^2} = \alpha T(p_1) + \beta T(p_2)$$
And so T is a linear transformation.

(b) (5 pts) Using $\{1, x, x^2\}$ as a basis for P_2, find the matrix representing T.

- Let $B = \{b_1, b_2, b_3\} \equiv \{1, x, x^2\}$ and let $i_B : P_2 \to \mathbb{R}^3$ be the corresponding coordinatization of P_2
$$\begin{aligned} T(b_1) &= T(1) = 0 = 0 \cdot 1 + 0 \cdot x + 0 \cdot x^2 \quad \Rightarrow \quad T(b_1)_B = [0, 0, 0] \\ T(b_2) &= T(x) = 0 = 0 \cdot 1 + 0 \cdot x + 0 \cdot x^2 \quad \Rightarrow \quad T(b_2)_B = [0, 0, 0] \\ T(b_3) &= 2x = 0 \cdot 1 + 2 \cdot x + 0 \cdot x^2 \quad \Rightarrow \quad T(b_1)_B = [0, 2, 0] \end{aligned}$$
So
$$\mathbf{A}_T = \left[\begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ T(b_1)_B & T(b_2)_B & T(b_3)_B \\ \downarrow & \downarrow & \downarrow \end{array}\right] = \left[\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{array}\right]$$

(c) (5 pts) What is the kernel of T?.

- It is clear the that solution set of $A_T\mathbf{x} = \mathbf{0}$ is $span([0, 0, 1]) \longleftrightarrow span([b_1]_B, [b_2]_B)$. Pulling back to polynomials, we have
$$\ker(T) = span(1, x)$$

(d) (5 pts) What is the image of T?

- It is clear that the column space of $\mathbf{A}_T$ is just $span([0, 1, 0]) = span([b_2]_B)$, and so
$$im(T) = span(b_2) = span(x)$$

4. (15 pts) Let S be a subspace of vector space V. Suppose $\{s_1, \ldots, s_k\}$ is a basis for S and $\{s_1, \ldots, s_k, v_{k+1}, \ldots, v_n\}$ is a basis for V. Find a basis for the quotient space V/S (that is, find a set of generators for V/S and verify that they are linearly independent.).

- Let $p_S : V \to V/S$ be the canonical projection of V onto V/S. Since p_S is surjective we have
$$\begin{aligned} V/S &= im(p_S) = \{p_S(v) \mid v \in V\} \\ &= \{p_S(a_1s_1) + \cdots + a_ks_k + a_{k+1}v_{k+1} + \cdots + a_nv_n \mid a_1, \ldots, a_n \in \mathbb{F}\} \\ &= \{\mathbf{0}_{V/S} + \cdots + \mathbf{0}_{V/S} + a_{k+1}p_S(v_{k+1}) + \cdots + a_np_S(v_n)\} \qquad \text{since each } a_is_i \in S \subset \ker(p_S) \\ &= span(p_S(v_{k+1}), \ldots, p_S(v_n)) \end{aligned}$$
Thus, V/S is generated by $p_S(v_{k+1}), \ldots, p_S(v_n)$. To see that these vectors are also linearly independent, suppose we had
$$a_{k+1}p_S(v_{k+1}) + \cdots + a_np_S(v_n) = \mathbf{0}_{V/S}$$
This then implies
$$\mathbf{0}_{V/S} = p_S(a_{k+1}v_{k+1} + \cdots + a_nv_n) \quad \Rightarrow \quad a_{k+1}v_{k+1} + \cdots + a_nv_n \in \ker(p_S) = S$$
But this would be a contradiction unless each coefficient $a_{k+1}, \ldots, a_k$ is equal to 0; for, by construction the vectors $v_{k+1}, \ldots, v_n$ live outside of the subspace S. Thus,
$$a_{k+1}p_S(v_{k+1}) + \cdots + a_np_S(v_n) = \mathbf{0}_{V/S} \quad \Rightarrow \quad 0 = a_{k+1} = a_{k+2} = \cdots = a_n$$
and so the vectors $p_S(v_{k+1}), \ldots, p_S(v_n)$ are linearly independent and hence constitute a basis for V/S.

5.

(a) (5 pts) Compute the determinant of $\mathbf{A} = \begin{pmatrix} 1 & 0 & 3 \\ 3 & 2 & 2 \\ 0 & 1 & 0 \end{pmatrix}$ using a cofactor expansion.

- Cofactor expansion along the last row:

$$\begin{aligned} \det(\mathbf{A}) &= (0)\cdot(-1)^{3+1}\,dt\begin{pmatrix} 0 & 3 \\ 2 & 2 \end{pmatrix} + (1)\cdot(-1)^{3+2}\det\begin{pmatrix} 1 & 3 \\ 3 & 2 \end{pmatrix} + (0)\cdot(-1)^{3+3}\det\begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix} \\ &= 0 + (-1)(2-9) + 0 \\ &= 7 \end{aligned}$$

b. (5 pts) Compute the determinant of $\mathbf{A} = \begin{pmatrix} 0 & 1 & 1 \\ 2 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix}$ using elementary row operations.

$$\det\begin{pmatrix} 0 & 1 & 1 \\ 2 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix} \xrightarrow{R_1 \longleftrightarrow R_3} -\det\begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \xrightarrow{R_2 \longleftrightarrow R_2 - 2R_1} -\det\begin{pmatrix} 1 & 2 & 1 \\ 0 & -3 & -2 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\xrightarrow{R_2 \longleftrightarrow R_2 + 3R_3} -\det\begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \xrightarrow{R_2 \longleftrightarrow R_3} +\det\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = 1$$

6. (10 pts) Apply Crammer's Rule to solve the following linear system.

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 4 \\ x_1 - x_2 + x_3 &= 0 \\ x_1 + x_2 - 3x_3 &= 0 \end{aligned}$$

- $$\det\mathbf{A} = \det\begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -3 \end{pmatrix} = 12$$

  , determinant: 12

$$\begin{aligned} \det\mathbf{B}_1 &= \det\begin{pmatrix} 4 & 2 & 1 \\ 0 & -1 & 1 \\ 0 & 1 & -3 \end{pmatrix} = 8 \\ \det(\mathbf{B}_2) &= \det\begin{pmatrix} 1 & 4 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & -3 \end{pmatrix} = -16 \\ \det(\mathbf{B}_3) &= \det\begin{pmatrix} 1 & 2 & 4 \\ 1 & -1 & 0 \\ 1 & 1 & 0 \end{pmatrix} = 8 \end{aligned}$$

  So

$$\begin{aligned} x_1 &= \frac{\det(\mathbf{B}_1)}{\det(\mathbf{A})} = \frac{2}{3} \quad , \quad x_2 = \frac{\det(\mathbf{B}_2)}{\det(\mathbf{A})} = -\frac{4}{3} \quad , \quad x_3 = \frac{\det(\mathbf{B}_3)}{\det(\mathbf{A})} = \frac{2}{3} \\ \mathbf{x} &= \left[\frac{2}{3}, -\frac{4}{3}, \frac{2}{3}\right] \end{aligned}$$

7.

(a) (10 pts) Compute the cofactor matrix $\mathbf{C}$ of $\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 3 & 0 & 0 \end{pmatrix}$.

$$\det(\mathbf{A}) = 6$$

$$\begin{aligned}
\det(M_{11}) &= \det\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = 0 \quad , \quad \det(M_{12}) = \det\begin{pmatrix} 0 & 2 \\ 3 & 0 \end{pmatrix} = -6 \quad , \quad \det(M_{13}) = \det\begin{pmatrix} 0 & 0 \\ 3 & 0 \end{pmatrix} = 0 \\
\det(M_{21}) &= \det\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 0 \quad , \quad \det(M_{22}) = \det\begin{pmatrix} 0 & 0 \\ 3 & 0 \end{pmatrix} = 0 \quad , \quad \det(M_{23}) = \det\begin{pmatrix} 0 & 1 \\ 3 & 0 \end{pmatrix} = -3 \\
\det(M_{31}) &= \det\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = 2 \quad , \quad \det(M_{32}) = \det\begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} = 0 \quad , \quad \det(M_{33}) = \det\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 0
\end{aligned}$$

$$\mathbf{C}_{ij} = (-1)^{i+j}\det(M_{ij}) \quad \Rightarrow \quad \mathbf{C} = \begin{pmatrix} 0 & 6 & 0 \\ 0 & 0 & 3 \\ 2 & 0 & 0 \end{pmatrix}$$

(b) (5 pts) Use the result of (a) to compute $\mathbf{A}^{-1}$.

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\mathbf{C}^T = \frac{1}{6}\begin{pmatrix} 0 & 0 & 2 \\ 6 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \frac{1}{3} \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix}$$