Math 4063-5023
SOLUTIONS TO FIRST EXAM
9:00 – 10:15 am, Oct 2, 2014

You must write your answers in complete sentences and full detail to receive full credit.

1. Definitions. Write down the precise definitions of the following notions. (5 pts each)

(a) *subspace*

- A subset S of a vector space V over a field $\mathbb{F}$ is a *subspace* if S is closed under both scalar multiplication and vector addition; i.e.,

$$\begin{aligned} \mathbf{s} &\in S \quad , \quad \lambda \in \mathbb{F} \quad \Rightarrow \quad \lambda\mathbf{s} \in S \\ \mathbf{s}, \mathbf{s}' &\in S \qquad\qquad \Rightarrow \quad \mathbf{s} + \mathbf{s}' \in S \end{aligned}$$

(b) *dependence relation.*

- A *dependence relation* amongst a set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is an valid equation of the form

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k = \mathbf{0}_V$$

where the coefficients $a_1, a_2, \ldots, a_k$ are elements of the underlying field with at least one $a_i \neq 0_{\mathbb{F}}$.

(c) *linearly independent set of vectors*

- A set of vectors is *linearly independent* if there are no dependence relations amongst the vectors.

(d) *span of a set of vectors*

- The *span* of a set of vectors $\{\mathbf{v}_1, \ldots \mathbf{v}_k\}$ is the set of all possible linear combinations of these vectors; i.e.,

$$span\,(\mathbf{v}_1, \ldots, \mathbf{v}_k) = \{a_1\mathbf{v}_1 + \cdots + a_k\mathbf{v}_k \mid a_1, \ldots, a_k \in \mathbb{F}\}$$

(e) *basis for a vector space*

- A basis for a vector space V is a set of linearly independent vectors spanning V.

(f) *$n \times m$ homogeneous linear system*

- A *homogeneous linear system* is a set of n linear equations in m unknowns of the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m &= 0_{\mathbb{F}} \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_m &= 0_{\mathbb{F}} \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nm}x_m &= 0_{\mathbb{F}} \end{aligned}$$

2. (15 pts) Suppose S and T are subspaces of a vector space V. Prove that $S \cap T$ is also a subspace of V.

- In view of Proposition 4 on the last page it suffices to show that $\mathbf{v}, \mathbf{u} \in S \cap T$ and $\lambda, \mu \in \mathbb{F}$ implies that $\lambda\mathbf{v} + \mu\mathbf{u} \in S \cap T$. Now $\mathbf{v}, \mathbf{u} \in S \cap T \quad \Rightarrow \quad \mathbf{v}, \mathbf{u} \in S$; and since S is a subspace, $\lambda\mathbf{v} + \mu\mathbf{u} \in S$ for all $\lambda, \mu \in \mathbb{F}$. Similarly, $\mathbf{v}, \mathbf{u} \in S \cap T \quad \Rightarrow \quad \mathbf{v}, \mathbf{u} \in T \quad \Rightarrow \quad \lambda\mathbf{v} + \mu\mathbf{u} \in T$. Finally,

$$\left.\begin{array}{l} \lambda\mathbf{v} + \mu\mathbf{u} \in S \\ \lambda\mathbf{v} + \mu\mathbf{u} \in T \end{array}\right\} \quad \Rightarrow \quad \lambda\mathbf{v} + \mu\mathbf{u} \in S \cap T$$

and we're done.

3. (20 pts) Prove that if S is a subspace of a finite dimensional vector space and $\dim(S) = \dim(V)$, then $S = V$.

- Let $\{\mathbf{s}_1, \dots, \mathbf{s}_n\}$ be a basis for S and $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be a basis for V (these two bases have the same cardinality since, by hypothesis, $\dim(S) = \dim(V)$). Since each $\mathbf{s}_i \in V$, we have
$$span(\mathbf{s}_1, \dots, \mathbf{s}_n) \subseteq V$$
If $span(\mathbf{s}_1, \dots, \mathbf{s}_n) \neq V$, then there must exist a non-zero vector $\mathbf{v} \in V$ that does not lie in $span(\mathbf{s}_1, \dots, \mathbf{s}_n)$. I claim $\{\mathbf{s}_1, \dots, \mathbf{s}_n, \mathbf{v}\}$ is a linearly indepdent set. For if we had a dependence relation
$$a_1\mathbf{s}_1 + \cdots + a_n\mathbf{s}_n + b\mathbf{v} = \mathbf{0}_V$$
then either (i) $b = 0_{\mathbb{F}}$ and we have a dependence relation amongs the basis vectors for S (which is impossible since basis vectors are linearly independent) or (ii) $b \neq 0_{\mathbb{F}}$ and we have an equation of the form
$$-\frac{1}{b}(a_1\mathbf{s}_1 + \cdots + a_n\mathbf{s}_n) = \mathbf{v}$$
But this can't happen either since the left hand side is in $span(\mathbf{s}_1, \dots, \mathbf{s}_n)$ but $\mathbf{v} \notin span(\mathbf{s}_1, \dots, \mathbf{s}_n)$. Therefore no such $\mathbf{v}$ can exist. We conclude
$$S = span(\mathbf{s}_1, \dots, \mathbf{s}_n) = V$$

4. (15 pts) Let
$$p_1 = 1 + x - 2x^2 + x^3 \quad , \quad p_2 = 1 - 2x^2 + 2x^3 \quad , \quad p_3 = x - x^3 \quad , \quad p_4 = 1 + 3x - 2x^2 - x^3 \qquad .$$
Find a basis for $S = span(p_1, p_2, p_3, p_4)$.

- $B = \{1, x, x^2, x^3\}$ is a basis for the polynomials of degree ≤ 3. With respect to this basis, the coordinate vectors for p_1, p_2, p_3, and p_4 are
$$p_1 \longleftrightarrow 1, 1, -2, 1 \quad , \quad p_2 \longleftrightarrow [1, 0, -2, 2] \quad , \quad p_3 \longleftrightarrow [0, 1, 0, -1] \quad , \quad p_4 \longleftrightarrow [1, 3, -2, -1]$$
and so the coefficient matrix (see Theorem 10 on the last page) for $[p_1, p_2, p_3, p_4]$ is
$$\begin{bmatrix} 1 & 1 & -2 & 1 \\ 1 & 0 & -2 & 2 \\ 0 & 1 & 0 & -1 \\ 1 & 3 & -2 & -1 \end{bmatrix} \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 1 & -2 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 2 & 0 & -2 \end{bmatrix} \xrightarrow[R_4 \to R_4 + 2R_2]{R_3 \to R_3 + R_2} \begin{bmatrix} 1 & 1 & -2 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
Evidently, the row space of the coefficient matrix is spanned by the two linearly independent vectors $[1, 1, -2, 1]$ and $[0, -1, 0, 1]$. The corresponding polynomials $1 + x - 2x^2 + x^3$ and $-x + x^3$ will be a basis for $span(p_1, p_2, p_3, p_4.)$:
$$span(p_1, p_2, p_3, p_4) = span(p_1, p_3)$$

5. (15 pts) Find a basis for the solution set of the following homogeneous linear system.

$$\begin{aligned} x_1 - x_3 + 2x_4 &= 0 \\ x_2 + x_3 - x_4 &= 0 \\ 2x_1 + x_2 - x_3 + 3x_4 &= 0 \end{aligned}$$

- First we row-reduce the augmented matrix this system is to reduced row echelon form:

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & 2 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 2 & 1 & -1 & 3 & 0 \end{array}\right] \quad \rightarrow \quad \left[\begin{array}{cccc|c} 1 & 0 & -1 & 2 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The first two columns of the RREF contain the its pivots, so variable x_1 and x_2 will be regarded as the variables-we-can-solve-for. That leaves x_3 and x_3 as free parameters. Writing down the equations corresponding to matrix in RREF and moving the free parameters to the right hand side yields

$$\left.\begin{array}{r} x_1 - x_3 + 2x_4 = 0 \\ x_2 + x_3 - x_4 = 0 \\ 0 = 0 \end{array}\right\} \quad \Rightarrow \quad \begin{array}{c} x_1 = x_3 - 2x_4 \\ x_2 = -x_3 + x_4 \end{array}$$

Therefore, a general solution vector will be

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_3 - 2x_4 \\ -x_3 + x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

Therefore, $\{[1,-1,1,0], [-2,1,0,1]\}$ will provide a basis for the solution space of the homogeneous linear system.

6. (20 pts) Use Lemma 14 (on last page) to prove that an $n \times m$ linear system $\mathbf{Ax} = \mathbf{0}$ has a nontrivial solution if and only if $rank(\mathbf{A}) < m$.

- Let $[\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_m]$ be the columns of $\mathbf{A}$ and let $\mathbf{x} = [x_1, \ldots, x_m]$ be a solution of $\mathbf{Ax} = \mathbf{0}$, then by Lemma 14 we have

$$x_1\mathbf{c}_1 + x_2\mathbf{c}_2 + \cdots + x_m\mathbf{c}_m = \mathbf{0}$$

This would constitute a dependence relation amongst the column vectors if any of the components of $\mathbf{x}$ are non-zero. Thus, we can have non-trivial solutions only if the columns of $\mathbf{A}$ are linearly dependent. This would, in turn, imply that

$$\text{non-trivial solution} \quad \Rightarrow \quad rank(\mathbf{A}) \equiv \dim(ColSp(\mathbf{A})) < \#\text{columns} = m$$

On the other hand, if $rank(\mathbf{A}) < m$, then the columns of $\mathbf{A}$ are linearly dependent, and so there is a dependence relation of the form

$$a_1\mathbf{c}_1 + \cdots + a_m\mathbf{c}_m = \mathbf{0}$$

with not all $a_i = 0_{\mathbb{F}}$. But then setting $\mathbf{x} = [a_1, \ldots, a_m]$ we would have a non-trivial solution of $\mathbf{Ax} = \mathbf{0}$.

Basic Theorems

The statements listed below you can cite and use without proof in your solutions.

Proposition 1. *The zero vector $\mathbf{0}_{\mathbb{V}}$ of a vector space is unique.*

Proposition 2. *Let V be a vector space over a field $\mathbb{F}$. Then $0_F \cdot v = \mathbf{0}_V$ for all $v \in V$.*

Proposition 3. *If S is a subspace of a vector space V , then $\mathbf{0}_V \in S$.*

Proposition 4. *A subset S of a vector space V over a field $\mathbb{F}$ is a subspace if and only if every linear combination of the form $\alpha v + \beta u$ with $\alpha, \beta \in \mathbb{F}$, $v, u \in S$ is in S.*

Proposition 5. *$span\left(v_1, \ldots v_{k+1}\right) = span\left(v_1, \ldots, v_k\right)$ if and only $v_{k+1} \in span\left(v_1, \ldots, v_k\right)$.*

Theorem 6. *Let S be a subspace of a vector space V over a field $\mathbb{F}$. Suppose S is generated by n vectors $v_1, \ldots, v_n$. Let $\{w_1, \ldots, w_m\}$ be a set of m vectors in S with $m > n$. Then the vectors $\{w_1, \ldots, w_m\}$ are linearly dependent.*

Corollary 7. *Suppose $\{v_1, \ldots, v_n\}$ and $\{w_1, \ldots, w_m\}$ are two bases for a subspace S. Then $n = m$.*

Proposition 8. *Let $[v_1, \ldots, v_n]$ be an $n \times m$ matrix. If an elementary operation (see Corollary 4.2 and Definition 4.3) is applied to this list of vectors, the new list of vectors is matrix that has the same row space. More generally, if $\mathbf{M}$ is a matrix and $\mathbf{M}'$ is a matrix obtained from $\mathbf{M}$ by applying a sequence of elementary row operations to the (row) vectors of $\mathbf{M}$ (and the intermediary matrices). Then*

$$RowSp\left(\mathbf{M}'\right) = RowSp\left(\mathbf{M}\right)$$

Proposition 9. *Let $\mathbf{A}$ be an $n \times m$ matrix. Then there exists a sequence of elementary operations that converts $\mathbf{A}$ to a matrix in row echelon form.*

Theorem 10. *Let V be an m-dimensional vector space with basis $B = [v_1, \ldots, v_m]$ and $\mathbf{A}$ be the coefficient matrix of a set of n non-zero vectors $[u_1, \ldots, u_n]$ with respect to B. Suppose that the row vectors $\mathbf{r}_1, \ldots, \mathbf{r}_n \in \mathbb{F}^m$ of $\mathbf{A}$ are in row echelon form. Then the vectors $u_1, \ldots, u_n$ are linearly independent.*

Lemma 11. *If $\{v_1, \ldots, v_m\}$ is a linearly dependent set and if $\{v_1, \ldots, v_{m-1}\}$ is a linearly independent set then v_m can be expressed as a linear combination of $v_1, \ldots, v_{m-1}$.*

Theorem 12. *Every finitely generated vector space has a basis.*

Theorem 13. *Let $V = span_{\mathbb{F}}\left(v_1, \ldots, v_m\right)$ be a finitely generated vector space. Then a basis for V can be selected from among the set of generators $\{v_1, \ldots, v_m\}$. In other words, any set of generators for a finitely generated vector space V contains a basis for V.*

Lemma 14. *Suppose $\mathbf{A}$ is an $n \times m$ matrix with column vectors $[\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_m]$ and $\mathbf{x}$ is a $n \times 1$ column vector with entries $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}$. Then*

$$\mathbf{A}\mathbf{x} = x_1\mathbf{c}_1 + x_2\mathbf{c}_2 + \cdots + x_m\mathbf{c}_m \quad .$$

Theorem 15. *Consider a $n \times m$ linear system with coefficient matrix $\mathbf{A}$ and inhomogenous part $\mathbf{b} \in \mathbb{F}^n$. For each i between 1 and n, let $\mathbf{c}_i$ denote the element of $\mathbb{F}^n$ formed by writing the entries in the i^{th} column of $\mathbf{A}$ in order (from top to bottom). Then the linear system has a solution if and only if either of the following two conditions is satisfied.*

(i) $\mathbf{b} \in span\left(\mathbf{c}_1, \ldots, \mathbf{c}_m\right)$
(ii) $\dim span\left(\mathbf{c}_1, \ldots, \mathbf{c}_m\right) = \dim span\left(\mathbf{c}_1, \ldots, \mathbf{c}_m, \mathbf{b}\right)$

Proposition 16. *Let $\mathbf{x}$ be a solution of an $n \times m$ linear system $S\left(\mathbf{A}, \mathbf{b}\right)$, and let S be the solution set of the corresponding homogeneous linear system $S\left(\mathbf{A}, \mathbf{0}\right)$. Then the solution set of $S\left(\mathbf{A}, \mathbf{b}\right)$ coincides with the hyperplane through $\mathbf{x}$ generated by S.*

Theorem 17. *The reduced row echelon form of an $n \times m$ matrix $\mathbf{A}$ is unique.*

Proposition 18. *If $[\mathbf{B} \mid \mathbf{c}]$ is a matrix in reduced row echelon form obtained from $[\mathbf{A} \mid \mathbf{b}]$ by a sequence of elementary row operations, then the solutions to the linear system corresponding to $[\mathbf{A} \mid \mathbf{b}]$ will be the same as the solutions to the linear system corresponding to $[\mathbf{B} \mid \mathbf{c}]$.*

Theorem 19. *Let $\mathbf{A}$ be an $n \times m$ matrix, then the row space of $\mathbf{A}$ equals the dimension of its column space.*