LECTURE 12

Homomorphisms and Isomorphisms

While I have discarded some of Curtis's terminology (e.g. "linear manifold") because it served mainly to reference something (differential geometry) that is esoteric to the present course; I now find myself wanting to break from the text in the other direction; wanting to discard the nomenclature "linear transformation" in favor of a notion of wider applicability.

The basic idea of a **homomorphism** is that it is a mapping that keeps you in the same category of objects and is compatible with the basic structural operations on such objects. For example, a ring homomorphism is a mapping between rings that is compatible with the ring properties of the domain and codomain, a group homomorphism is a mapping between groups that is compatible with the group multiplication in the domain and codomain. In this course, we have defined linear transformations as mappings that are compatible with the vector space properties of the domain and codomain (specifically, mappings that are compatible with scalar multiplication and vector addition). For this reason, allow me now to shift into a more modern parlance and refer to linear transformations as *vector space homomorphisms*.

Let $T : V \to W$ be a vector space homomorphism. Let me recall a bit of Lecture 10. Attached to T we have two important subspaces:

$$\begin{aligned} \ker T &= \{v \in V \mid T(v) = \mathbf{0}_W\} \text{ is a subspace of } V \\ range(T) &= \{w \in W \mid w = T(v) \text{ for some } v \in V\} \text{ is a subspace of } W \end{aligned}$$

These are important because

$$\begin{aligned} \ker(T) &= \{\mathbf{0}_V\} \quad \Rightarrow \quad T \text{ is injective (one-to-one)} \\ range(T) &= W \quad \Rightarrow \quad T \text{ is surjective (onto)} \end{aligned}$$

and that if both these conditions hold T is bijective, and so has an inverse $T^{-1} : W \to V$ which is also a vector space homomorphism.

Definition 12.1. *If $T : V \to W$ is a vector space homomorphism such that $\ker(T) = \{\mathbf{0}_V\}$ and $range(T) = W$, then we say T is a **vector space isomorphism.***

Lemma 12.2. *Suppose $T : V \to W$ is a vector space isomorphism and $\{v_1, \dots, v_n\}$ is a basis for V. Then $\{T(v_1), \dots, T(v_n)\}$ is a basis for W.*

Proof. Let $B = \{v_1, \dots, v_m\}$ be a basis for V. I claim $\{T(v_1,), \dots, T(v_m)\}$ is a basis for W. First of all, $T = span(T(v_1), \dots, T(v_n))$ because

$$\begin{aligned} W &= Range(T) = \{T(v) \mid v \in V\} \\ &= \{T(a_1 v_1 + \cdots + a_m v_m) \mid a_1, \dots, a_m \in \mathbb{F}\} \\ &= \{a_1 T(v_1) + \cdots + a_m T(v_m) \mid a_1, \dots, a_m \in \mathbb{F}\} \\ &= span(T(v_1), \dots T(v_n)) \end{aligned}$$

Moreover, the vectors $\{T(v_1), \ldots, T(v_m)\}$ are linearly independent since

$$
\begin{aligned}
\mathbf{0}_W &= a_1 T(v_1) + \cdots + a_m T(v_m) \\
&= T(a_1 v_1 + \cdots + a_m v_m) \\
&\Rightarrow \quad a_1 v_1 + \cdots + a_m v_m \in \ker(T) \\
&\Rightarrow \quad a_1 v_1 + \cdots + a_m v_m = \mathbf{0}_V \\
&\Rightarrow \quad a_1 = 0_{\mathbb{F}}, \ldots, a_m = 0_{\mathbb{F}} \\
&\Rightarrow \quad T(v_1), \ldots, T(v_m) \text{ are linearly independent.}
\end{aligned}
$$

We have thus shown that $\{T(v_1), \ldots, T(v_n)\}$ is a set of linearly independent generators of W; hence $\{T(v_1), \ldots, T(v_n)\}$ is a basis for W. □

COROLLARY 12.3. *If $T : V - \to W$ is a vector space isomorphism between two finitely generated vector spaces, then* $\dim(V) = \dim(W)$.

Proof. Since the number of vectors in this basis for W is equal to the number of vectors in basis for V, the dimensions of V and W must also be the same. □

Recall that once we adopt a basis $B = \{v_1, \ldots, v_m\}$ for a vector space V we have the following *coordinatization* isomorphism

$$
i_B : V \to \mathbb{F}^m \quad : \quad v = a_1 v_1 + \cdots + a_m v_m \longmapsto [a_1, \ldots, a_m] \in \mathbb{F}^m
$$

In the preceding lecture we used such isomorphims to attach to a homomorphism $T : V \to W$ between two vector spaces a particular matrix $A_{T,B,B'}$. Here's how that worked. Along with the basis B for V, chooose a basis $B' = \{w_1, \ldots, w_n\}$ for W. The corresponding maps i_B and $i_{B'}$ will allow us to represent vectors in our abstract vector spaces V and W as concrete, calculable, vectors in, respectively, $\mathbb{F}^m$ and $\mathbb{F}^n$. Let me write

$$
\begin{aligned}
\mathbf{v}_B &= i_B\ (v) = \text{“coordinates of } v\text{” in } \mathbb{F}^m \\
\mathbf{w}_{B'} &= i_{B'}(w) = \text{”coordinates of } w\text{” in } \mathbb{F}^n
\end{aligned}
$$

We can now attach to $T : V \to W$ the $n \times m$ matrix $\mathbf{A}_{T,B,B'}$ defined by

$$
\mathbf{A}_{T,B,B'} = \left(\begin{array}{ccc} \uparrow & & \uparrow \\ i_{B'}(T(v_1)) & \cdots & i_{B'}(T(v_m)) \\ \downarrow & & \downarrow \end{array} \right)
$$

where we are to express each $i_{B'}(T(v_i)) \in \mathbb{F}^n$ as a column vector.

THEOREM 12.4. *Retain the setup of the preceding paragraph.*

$$
i_{B'}(T(v)) = \mathbf{A}_{T,B,B'} \mathbf{v}_B \quad \textit{for all } v \in V \quad .
$$

In other words the coordinates of the image $T(v)$ of v, can be obtained directly from the coordinates $\mathbf{v}_B$ of v by multiplying $\mathbf{v}_B$ by the matrix $\mathbf{A}_{T,B,B'}$.

Proof. Exploiting the basis B we can always write

$$
v = a_1 v_1 + \cdots + a_m v_m
$$

(Note that this means $\mathbf{v}_B = [a_1, \ldots, a_m]$.) Applying T we get

$$
\begin{aligned}
T(v) &= T(a_1 v_1 + \cdots + a_m v_m) \\
&= a_1 T(v_1) + \cdots + a_n T(v_m) \qquad \text{since } T \text{ is a linear transformation}
\end{aligned}
$$

Now apply (the linear transformation) $i_{B'}$ to both sidesw

$$\begin{aligned} i_{B'}(T(v)) &= i_{B'}(a_1 T(v_1) + \cdots + a_m T(v_m)) \\ &= a_1 i_{B'}(T(v_1)) + \cdots + a_m i_{B'}(T(v_m)) \\ &= a_1 \left(1^{st} \text{ column of } \mathbf{A}_{T,B,B'}\right) + \cdots + a_m \left(m^{th} \text{ column of } \mathbf{A}_{T,B,B'}\right) \\ &= \begin{pmatrix} \uparrow & & \uparrow \\ i_{B'}(T(v_1)) & \cdots & i_{B'}(T(v_m)) \\ \downarrow & & \downarrow \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix} \\ &= \mathbf{A}_{T,B,B'} \mathbf{v}_B \end{aligned}$$

□

COROLLARY 12.5. *Let $T : V \to W$ be a linear transformation between two finitely generated vector spaces, let B and B' be bases for, respectively, V and W, and let $\mathbf{A}_{T,B,B'}$ be the corresponding matrix (as constructed above). Then*

(i) $i_{B'}(Range(T)) = ColSp(\mathbf{A}_{T,B,B'})$
(ii) $i_B(Ker(T)) = NullSp(\mathbf{A}_{T,B.B'})$

Proof. (i) The first statement is merely an observation from the proof of the preceding Theorem. For any $v \in V$ we could write $v = a_1 v_1 + \cdots a_m v_m$, $v \in B$, and then

$$i_{B'}(T(v)) = a_1 \left(1^{st} \text{ column of } \mathbf{A}_{T,B,B'}\right) + \cdots + a_m \left(m^{th} \text{ column of } \mathbf{A}_{T,B,B'}\right)$$

Letting v run over V, which means letting the coefficients $a_1 \ldots, a_m$ run over the underlying field, we see that

$$\{i_{B'}(T(v)) \mid v \in V\} = span\,(\text{columns of } \mathbf{A}_{T,B'B'})$$

or, equivalently

$$i_{B'}(Range(T)) = ColSp(\mathbf{A}_{T,B,B'})$$

□

(ii) We have

$$\begin{aligned} i_B(Ker(T)) &= \{i_B(v) \mid v \in Ker(T)\} \\ &= \{i_B(v) \mid T(v) = \mathbf{0}_W\} \qquad (*) \end{aligned} \tag{12.1}$$

Since $i_{B'}$ is an isomorphism

$$T(v) = \mathbf{0}_W \iff i_{B'}(T(v)) = i_{B'}(\mathbf{0}_W) = \mathbf{0}_{\mathbb{F}^n}$$

But then, by the preceding theorem,

$$i_{B'}(T(v)) = \mathbf{A}_{T,B,B'} \mathbf{v}_B = \mathbf{0}_{\mathbb{F}^n}$$

Thus,

$$\{i_B(v) \mid T(v) = \mathbf{0}_W\} = \{\mathbf{v} \in \mathbb{F}^m \mid \mathbf{A}_{T,B,B'}\mathbf{v} = \mathbf{0}_{\mathbb{F}^n}\} = NullSp(\mathbf{A}_{T,B,B'})$$

and so (ii) follows. □

THEOREM 12.6. *Let $T : V \to W$ be a vector space homomorphism between two finitely generated vector spaces. Then*

$$\dim(V) = \dim(range(T)) + \dim(\ker(T))$$

Proof. In this proof, we simply continue to exploit the connection between linear transformations and their representative matrices. We have

$$\begin{aligned} \dim(V) &= \#\text{ basis vectors for } V \\ &= \#\text{ columns of } \mathbf{A}_{T,B,B'} \end{aligned}$$

On the other hand, since $i_{B'}$ is an isomorphism, by (i) of Corollary 12.4,

$$\begin{aligned}\dim\left(Range\left(T\right)\right) &= \dim\left(ColSp\left(\mathbf{A}_{T,B,B'}\right)\right)\\ &= \#\text{ pivots in any row echelon form of }\mathbf{A}_{T,B,B'}\end{aligned}$$

On yet another hand, since i_B is an isomorphism

$$\begin{aligned}\dim\left(Ker\left(T\right)\right) &= \dim\left(NullSp\left(\mathbf{A}'_{T,B,B}\right)\right)\\ &= \dim\left(\text{ solution set of }\mathbf{A}_{T,B,B'}\mathbf{x}=\mathbf{0}\right)\\ &= \#\text{ free parameters in solution set }\mathbf{A}_{T,B,B'}\mathbf{x}=\mathbf{0}\\ &= \#\text{ columns of a row echelon form of }\mathbf{A}_{T,B,B'}\text{ that lack pivots}\end{aligned}$$

Thus, if $\widetilde{\mathbf{A}}_{T,B,B'}$ is any row echelon form of $\mathbf{A}_{T,B,B'}$, then

$$\begin{aligned}\dim\left(V\right) &= \#\text{ columns of }\mathbf{A}_{T,B,B'}\\ &= \#\text{ columns of }\widetilde{\mathbf{A}}_{T,B,B'}\\ &= \#\left\{\text{columns of }\widetilde{\mathbf{A}}_{T,B,B'}\text{with pivots}\right\}+\#\left\{\text{columns of }\widetilde{\mathbf{A}}_{T,B,B'}\text{withot pivots}\right\}\\ &= \dim\left(Range\left(T\right)\right)+\dim\left(Ker\left(T\right)\right)\end{aligned}$$

□

1. Calculating T^{-1}

It should be no surprise that in order to calculate the inverse of a vector space isomorphism $T : V \to W$, we will need to calculate the inverse of an associated matrix $\mathbf{T}_{B,B'}$.

However, we have yet to discuss matrix inversion. Let's take care of that topic straight away.

1.1. Calculating Matrix Inverses.

1.1.1. *Elementary Row Operations and Matrix Multiplication.*

Lemma 12.7. *Each elementary row operation on an $n \times m$ matrix $\mathbf{M}$ is implementable by a certain matrix multiplication by a $n \times n$ matrix.*

Proof.

(i) The operation where you interchange the i and j^{th} row of $\mathbf{M}$ can be reproduced by multiplying $\mathbf{M}$ from the left by a matrix $\mathcal{E}_{i\to j}$ which is formed by interchanging the i^{th} and j^{th} row of the $n \times n$ identity matrix.

(ii) The operation of multiplying the i^{th} row of $\mathbf{M}$ by a scalar λ can be reproduced by multiplying the matrix $\mathbf{M}$ by a matrix $\mathcal{E}_\lambda$ which is formed by multiplying the i^{th} row of the $n \times n$ identity matrix by λ.

(iii) The operation of adding λ times the j^{th} row of $\mathbf{M}$ to the i^{th} row of $\mathbf{M}$ can be reproduced by multiplying the matrix $\mathbf{M}$ by a matrix $\mathcal{E}_{\imath+\lambda j}$ formed by adding λ times the j^{th} row of the $n \times n$ identity matrix to the i^{th} row of that identity matrix.

1.1.2. *An Algorithm to Calculate* $\mathbf{A}^{-1}$. Suppose $\mathbf{A}$ is row equivalent to the identity matrix. Let $\mathcal{R}_1, \mathcal{R}_2, \dots, \mathcal{R}_k$ be a sequence of elementary row operations that systematically transform $\mathbf{A}$ to the identity matrix:

$$\mathbf{A}\to\mathcal{R}_1\left(\mathbf{A}\right)\to\mathcal{R}_2\left(\mathcal{R}_1\left(\mathbf{A}\right)\right)\to\cdots\to\mathcal{R}_k\left(\mathcal{R}_{k-1}\left(\cdots\left(\mathcal{R}_2\mathcal{R}_1\left(\mathbf{A}\right)\right)\right)\right)=\mathbf{I}_n$$

Each of these row operations can also be implemented by matrix multiplications as in the Lemma above. Thus

$$\mathbf{A}\to\mathcal{E}_{\mathcal{R}_1}\mathbf{A}\to\mathcal{E}_{\mathcal{R}_2}\mathcal{E}_{\mathcal{R}_1}\mathbf{A}\to\cdots\to\mathcal{E}_{\mathcal{R}_k}\mathcal{E}_{\mathcal{R}_{k-1}}\cdots\mathcal{E}_{\mathcal{R}_2}\mathcal{E}_{\mathcal{R}_1}\mathbf{A}=\mathbf{I}_n$$

Thus,

$$\mathcal{E}_{\mathcal{R}_k}\mathcal{E}_{\mathcal{R}_{k-1}}\cdots\mathcal{E}_{\mathcal{R}_2}\mathcal{E}_{\mathcal{R}_1} = \mathbf{A}^{-1}$$

Equivalently, we could write

$$\mathbf{A}^{-1} = \mathcal{E}_{\mathcal{R}_k}\mathcal{E}_{\mathcal{R}_{k-1}}\cdots\mathcal{E}_{\mathcal{R}_2}\mathcal{E}_{\mathcal{R}_1}\mathbf{I}_n$$

or

$$\mathbf{A}^{-1} = \mathcal{R}_k\left(\mathcal{R}_{k-1}\left(\cdots\left(\mathcal{R}_2\mathcal{R}_1\left(\mathbf{I}_n\right)\right)\right)\right)$$

In other words, the same sequence of elementary row operations that converts $\mathbf{A}$ to the identity matrix $\mathbf{I}_n$ will convert $\mathbf{I}_n$ to the inverse of $\mathbf{A}$.

So here's what you do:

(i) Form an augmented matrix $[\mathbf{A}|\mathbf{I}_n]$.
(ii) Apply elementary row operations that transform the left hand side to reduced row echelon form.

$$[\mathbf{A}|\mathbf{I}_n] \quad \rightarrow \quad [\mathbf{A}'|\mathbf{B}]$$

(iii) Suppose that reduced row echelon form $\mathbf{A}'$ of $\mathbf{A}$ coincides with the identity matrix $\mathbf{I}_n$, then the matrix $\mathbf{B}$ on the right will have to be $\mathbf{A}^{-1}$. On the other hand, if $\mathbf{A}' \neq \mathbf{I}_n$, then that will mean that the matrix $\mathbf{A}$ is not invertible. (Note that if $\mathbf{A}' \neq \mathbf{I}_n$, then we'll have to have $rank\left(\mathbf{A}\right) < n$, which will mean that the map $T_{\mathbf{A}} : \mathbb{F}^n \rightarrow \mathbb{F}^n$ corresponding to matrix multiplication by $\mathbf{A}$ will not be surjective, and so not invertible.)

Example 12.8. Compute the inverse of $\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}$

- $$\begin{aligned}
[\mathbf{A}|\mathbf{I}_3] &= \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right) \\
&\xrightarrow{R_2 \rightarrow R_2 + R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right) \\
&\xrightarrow{R_3 \rightarrow R_3 - 2R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & -2 & -2 & 1 \end{array}\right) \\
&\xrightarrow{R_3 \rightarrow -R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 & 2 & -1 \end{array}\right) \\
&\xrightarrow{R_1 \rightarrow R_1 - R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & -2 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 & 2 & -1 \end{array}\right) \\
&\xrightarrow{R_2 \rightarrow R_2 - R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & -2 \\ 0 & 1 & 0 & -1 & -1 & 2 \\ 0 & 0 & 1 & 2 & 2 & -1 \end{array}\right)
\end{aligned}$$

  Therefore,

  $$\mathbf{A}^{-1} =: \begin{pmatrix} -1 & -2 & 1 \\ -1 & -1 & 1 \\ 2 & 2 & -1 \end{pmatrix}$$

  Indeed,

  $$\begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 & -2 \\ -1 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

1.1.3. *Inverting a Vector Space Isomorphism.*

THEOREM 12.9. *Suppose V is a vector space with basis B, W is a vector space with basis B' and $T : V \longrightarrow W$ is a vector space isomorphism from V to W. Then the inverse isomorphism $T^{-1} : W \longrightarrow V$ is given by*

$$T^{-1} = i_B^{-1} \circ T_{\mathbf{T}^{-1}_{B,B'}} \circ i_{B'}$$

where $i_B : V \to \mathbb{F}^n$ and $i_{B'} : W \to \mathbb{F}^n$ are the coordinazation maps corresponding to the bases B and B', and $T_{\mathbf{T}^{-1}_{B,B'}}$ is the linear transformation from $\mathbb{F}^n$ to $\mathbb{F}^n$ corresponding to matrix multiplication by the matrix inverse of $\mathbf{T}_{B,B'}$.

REMARK 12.10. This is just a formal result that exhibits explicitly the inverse linear transformation corresponding to a vector space isomorphism between two abstract vector spaces. For calculations, it much better to simply relate the coordinate vectors of V and W via the matrix $\mathbf{T}_{B,B'}$ and its inverse $\mathbf{T}^{-1}_{B'B'}$.