LECTURE 9

Hyperplanes

Let V be a finitely generated vector space over a field $\mathbb{F}$. Today we will focus our attention on a special class of subsets of V. These subsets will not in general be subspaces, but they arise quite naturally in linear algebra and have a lot of nice properties.

DEFINITION 9.1. *Let V be a vector space over a field $\mathbb{F}$, let S be a subspace of V and let $\mathbf{p} \in V$. Let*

$$M_{\mathbf{p},S} := \{\mathbf{p} + \mathbf{s} \mid \mathbf{s} \in S\}$$

*We shall refer to such a subset as a **linear submanifold** of V. The subspace S is called the **directing subspace** of $M_{\mathbf{p}_0,S}$. The **dimension** of $M_{\mathbf{p}_0,S}$ is defined as the dimension of S.*

REMARK 9.2. The notion of a manifold is actually more germane to differential geometry than linear algebra, in the geometric setting the notion of a linear manifold is akin to the notion of a linear function in Calculus - it's so simple that it's not worth discussing except as a simplifying limit.

However, because we **are not discussing** differential geometry in this course, I don't see much point in mentioning manifolds. What I think would be more helpful will be to view sets of the form $M_{\mathbf{p},S}$ as generalizations of lines and planes in $\mathbb{R}^3$. Indeed, you can generate a line in $\mathbb{R}^3$ by starting at a particular point $\mathbf{p} \in \mathbf{R}$ and then heading off an arbitrary distance d (forwards and backwards) along a particular direction $\mathbf{v}$: that is to say

$$line = \text{ set of the form } \left\{\mathbf{p} + t\mathbf{v} \mid \mathbf{p}, \mathbf{v} \in \mathbb{R}^3 \quad , \quad t \in \mathbb{R}\right\}$$

Since the vectors $\{t\mathbf{v} \mid t \in \mathbb{R}\}$ constitute the span of $\mathbf{v}$, such a line is a linear submanifold of $\mathbb{R}^3$ as defined defined above. Similarly, a plane in $\mathbb{R}^3$ is formed by starting at a particular point and then heading of an arbitary distance in two possible directrions; i.e. a subset of $\mathbb{R}^3$ of the form

$$\begin{aligned} plane &= \{\mathbf{p} + s\mathbf{u} + t\mathbf{v} \mid s, t \in \mathbb{R}\} \\ &= \{\mathbf{p} + \mathbf{s} \mid \mathbf{s} \in span\,(\mathbf{u}, \mathbf{v})\} \end{aligned}$$

To underscore this simple geometric picture, I shall henceforth refer to linear submanifolds of a vector space V as a *hyperplane* in V.

DEFINITION 9.3.

NOTATION 9.4 (common but abusive notation). Let V be a vector space. If S is a subspace of a vector space V, and $\mathbf{b}$ is a point of V we shall write

$$\mathbf{b} + S$$

for the corresponding hyperplane in V. (What's abusive about this is that you can't really add a subspace to a vector; on the other hand, if you interpret this expression as adding every vector in S to $\mathbf{b}$ then it does kind of make sense.)

I note also that we have already run into hyperplanes in two particular contexts. In Lecture 6, I defined quotient spaces V/S, S being some subspace of a vector space V, as the collection of sets of the form $\mathbf{p}_0 + S$ ($= [\mathbf{p}_0]_S$ in the notation of Lecture 6).

In Theorem 7.7 of Lecture 7, we saw hyperplanes arise as the solution sets of linear systems

THEOREM 7.7. (rephrased) *Suppose $Ax = b$ is an $n \times m$ linear system. Then the solution set of this linear system can be expressed as*

$$\mathbf{p}_0 + S = \{\mathbf{p}_0 + \mathbf{s} \mid \mathbf{s} \in S\}$$

where p_0 is any particular solution of $Ax = b$ and S is the solution set of $Ax = 0$.[1]

The first thing to point out about hyperplanes is that in general they are not subspaces. Here is a simple counter-example. Let $V = \mathbb{R}^2$, $\mathbf{p}_0 = [1, 0]$ and let $S = \mathit{span}\,([0, 1)]$. Then

$$\mathbf{p}_0 + S = \{[1, y] \mid y \in \mathbb{R}\}$$

which cannot be a subspace of $\mathbb{R}^2$ since it does not contain the zero vector in $\mathbb{R}^2$.

THEOREM 9.5. *Let $\mathbf{b} + S$ be a hyperplane in a vector space V. Then*

$$S = \{\mathbf{v} \in V \mid \mathbf{v} = \mathbf{r} - \mathbf{q} \quad , \quad \mathbf{r}, \mathbf{q} \in \mathbf{b} + S\}$$

Proof. Let

$$\widetilde{S} = \{\mathbf{v} \in V \mid \mathbf{v} = \mathbf{r} - \mathbf{q} \quad , \quad \mathbf{r}, \mathbf{q} \in \mathbf{b} + S\}$$

Suppose $\mathbf{v} \in \widetilde{S}$. Then there exists $\mathbf{r} = \mathbf{b} + \mathbf{s}_1$ and $\mathbf{q} = \mathbf{b} + \mathbf{s}_2$ in $\mathbf{b} + S$ such that

$$\mathbf{v} = \mathbf{r} - \mathbf{q} = (\mathbf{b} + \mathbf{s}_1) - (\mathbf{b} + \mathbf{s}_2) = \mathbf{s}_1 - \mathbf{s}_2 \in S$$

and so every element $\mathbf{v} \in \widetilde{S}$ is also a vector of S.

Suppose on the other hand that $\mathbf{s} \in S$, then we can always write

$$\mathbf{s} = \mathbf{s} + \mathbf{b} - \mathbf{b} = (\mathbf{b} + \mathbf{s}) - (\mathbf{b} + \mathbf{0})$$

which displays $\mathbf{s}$ as an element of $\widetilde{S}$. We conclude $\widetilde{S} = S$ and thus prove the theorem. □

Theorem 9.2 told us how to view the solution set of homogeneous $n \times m$ linear system is subspace of $\mathbb{F}^m$. The following lemma provides a converse to this result.

LEMMA 9.6. *Let S be an r-dimensional subspace of a vector space V of dimension m. Then there exists a set of $m - r$ homogeneous linear equations in m unknowns whose solution set is exactly S.*

Let $\{b_1, \ldots, b_r\}$ be a basis for S. Consider the solution space S^* of

$$\begin{aligned} b_1 \cdot x &= 0 \\ b_2 \cdot x &= 0 \\ &\vdots \\ b_r \cdot x &= 0 \end{aligned}$$

We first note that S^* is not likely to coincide with S, simply because for example, $b_1 \in S$ but $b_1 \cdot b_1 \neq 0$. On the other hand, since the vectors b_i are all linearly independent, it follows that the coefficient matrix $\mathbf{A}$ for this linear system has rank r (since the row space of $\mathbf{A}$ will be span of the r linearly independent vectors $b_1, \ldots, b_r$). So the solution space S^* of $\mathbf{A}x = \mathbf{0}$ will be of dimension $m - r$. Let $\{c_1, \ldots, c_{m-r}\}$ be the basis for the S^* and consider the system

$$\begin{aligned} c_1 \cdot x &= 0 \\ &\vdots \\ c_{m-r} \cdot x &= 0 \end{aligned} \qquad (*)$$

Let S^{**} denote the solution set of (*). Clearly, each b_i will be a solution of this system, and thus so will any linear combination of the vectors b_i, and thus, the entire subspace S lie in the solution set of (*). On the other hand, Since the vectors $c_1, \ldots, c_{m-r}$ are linearly independent, it is clear that the rank of this

[1]That the solution set of a homogeneous $n \times m$ linear system $\mathbf{Ax} = \mathbf{0}$ is actually a *subspace* of $\mathbb{F}^m$ is the content of Theorem 9.2.

linear system is $m-r$ and so solution set of dimension $m-(m-r)=r$. But we've seen that if a subspace has same dimension as the vector space containing it, the subspace must be the whole vector space. Since $S\subset S^{**}$ and $\dim(S)=\dim(S^{**})$ we conclude that S coincides with the solution set S^{**} of (*). $\square$

EXAMPLE 9.7. Find a homgeneous linear system whose solution set coincides with the span of $[1,0,1]$ and $[1,1,0]$.

- We first find a basis for the solution set of

$$\begin{aligned} 0 &= [1,0,1]\cdot x = x_1+x_3 \\ 0 &= [1,1,0]\cdot x = x_1+x_2 \end{aligned}$$

The augmented matrix for this system is

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{array}\right] \quad\rightarrow\quad \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$$

and so the general solution will be

$$\begin{aligned} x_1 &= -x_3 \\ x_2 &= x_3 \end{aligned}$$

or

$$\mathbf{x}=\left[\begin{array}{c} -x_3 \\ x_3 \\ x_2 \end{array}\right]=x_3\left[\begin{array}{c} -1 \\ 1 \\ 1 \end{array}\right]$$

Thus, the solution space has basis $c_1=[-1,1,1]$. The desired homogeneous linear system will be

$$[-1,1,1]\cdot x=0 \quad .$$

THEOREM 9.8. *A necessary and sufficient condition for a subset M of vectors to form a hyperplane in $\mathbb{F}^m$ of dimension r is that M be the set of solutions of a system of $m-r$ equations in m unknowns whose coefficient matrix has rank r.*

Proof. How, a solution set of a linear system constitutes a hyperplane was explained in at the start of this lecture. To see that every hyperplane $\mathbf{b}+S$ corresponds to a linear system, we just observe that by Lemma 10.5 the directing subspace S can be viewed as the solution set an $(m-r)\times m$ linear system $\mathbf{Ax}=\mathbf{0}$. Let

$$\widetilde{\mathbf{b}}=\mathbf{Ab} \quad .$$

Then any $\mathbf{b}+\mathbf{s}$ vector in $\mathbf{b}+S$ will satisfy

$$\mathbf{A}(\mathbf{b}+\mathbf{s})=\mathbf{Ab}+\mathbf{0}=\widetilde{\mathbf{b}} \quad .$$

This shows that the solution of

(**) $$\mathbf{Ay}=\widetilde{\mathbf{b}}$$

will contain $\mathbf{b}+S$. On the other hand, by construction $\mathbf{y}=\mathbf{b}$ is a solution of (**) and by Theorem 7.7, any other solution of $\mathbf{Ay}=\widetilde{\mathbf{b}}$ will be of the form

$$\mathbf{b} \quad + \text{some solution of } \mathbf{Ax}=0$$

and so any solution $\mathbf{y}$ of (**) will be of the form ,

$$\mathbf{y}=\mathbf{b}+\mathbf{s} \quad , \quad \mathbf{s}\in S \quad .$$

Therefore $\mathbf{b}+S$ will coincide with the solutions of (**). $\square$

Finally, let me describe an algorithm by which one can identify a linear system whose solution set is a given hyperplane.

We have see above that if we had a hyperplane in $\mathbb{R}^m$ which is also a subspace S of $\mathbb{R}^m$, then we could construct a corresponding equation set as follows:

- find a basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ for S
- find a basis $\{\mathbf{u}_1, \ldots, \mathbf{u}_\ell\}$ for the solution set of the linear system

$$\begin{aligned} \mathbf{v}_1 \cdot \mathbf{x} &= 0 \\ \mathbf{v}_2 \cdot \mathbf{x} &= 0 \\ &\vdots \\ \mathbf{v}_k \cdot \mathbf{x} &= 0 \end{aligned}$$

- The equations that cut out the subspace S will

$$\begin{aligned} \mathbf{u}_1 \cdot \mathbf{x} &= 0 \\ \mathbf{u}_2 \cdot \mathbf{x} &= 0 \\ &\vdots \\ \mathbf{u}_\ell \cdot \mathbf{x} &= 0 \end{aligned}$$

Now suppose we have a hyperplane in $\mathbb{R}^m$ of the form

$$H = \mathbf{p}_0 + S \equiv \{\mathbf{p}_0 + \mathbf{s} \mid \mathbf{s} \in S\}$$

S being some subspace of $\mathbb{R}^m$. Suppose also that we have followed the algorithm above and found ℓ vectors $\mathbf{u}_1, \ldots, \mathbf{u}_\ell$ such that

$$\mathbf{s} \in S \quad \Longleftrightarrow \quad \mathbf{u}_i \cdot \mathbf{s} = 0$$

Then each vector in H will satisfy

$$\mathbf{u}_i \cdot (\mathbf{p}_0 + \mathbf{s}) = \mathbf{u}_i \cdot \mathbf{p}_0 + \mathbf{u}_i \cdot \mathbf{s} = \mathbf{u}_i \cdot \mathbf{p}_0 + 0 = \mathbf{u}_i \cdot \mathbf{p}_0 \qquad , \quad i = 1, \ldots \ell$$

And so the linear equations whose solution set is the hyperplane $H = \mathbf{p}_0 + S$ will be

$$\begin{aligned} \mathbf{u}_1 \cdot \mathbf{x} &= \mathbf{u}_1 \cdot \mathbf{p}_0 \\ \mathbf{u}_2 \cdot \mathbf{x} &= \mathbf{u}_2 \cdot \mathbf{p}_0 \\ &\vdots \\ \mathbf{u}_\ell \cdot \mathbf{x} &= \mathbf{u}_\ell \cdot \mathbf{p}_0 \end{aligned}$$