LECTURE 3

# Dimension and Bases

In the preceding lecture, we introduced the notion of a subspace of a vector space and an easy way to construct subspace; namely, by considering the set of all possible linear combinations of a set of vectors: if $\{v_1, v_2, \dots, v_k\}$ is a set of vectors, then

$$span_{\mathbb{F}}(v_1, \dots, v_k) := \{\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k \mid \alpha_1, \alpha_2, \dots, \alpha_k \in \mathbb{F}\}$$

will be a subspace. We may also refer to $span_{\mathbb{F}}(v_1, \dots, v_k)$ as the **subspace generated by** $v_1, \dots, v_k$.

We noted that such a subspace may be generated may different ways; the exact same subspace being produced for different choices of $v_1, \dots, v_k$. In order to make the presentation of a vector in a subspace as simple as possible it made sense to try to work with as few generators as possible. This lead us to the following definition

Definition 3.1. *A set of vectors is* ***linearly dependent*** *if an equation of the form*

$$\alpha_1 v_1 + \cdots + \alpha_k v_k = \mathbf{0}_V$$

*can be satisfied without setting all the coefficients* $\alpha_1, \dots, \alpha_k$ *equal to* $0_{\mathbb{F}}$.

This definition is related to the problem of finding a minimal set of generators by

Proposition 3.2. $span_{\mathbb{F}}(v_1, \dots, v_k) = span_{\mathbb{F}}(v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_k)$ *if and only if there is a dependence relation amongst the vectors* $v_1, \dots, v_k$ *for which the coefficient* $\alpha_i$ *of* $v_i$ *is non-zero.*

So anytime there is a viable dependence relation we can toss out a generator without changing the nature of the subspace generated. When we there are no more viable dependence relations, this procedure terminates. The final minimal set of generators are then **linearly independent** since dependence relations amongst the vectors no longer exist.

We now turn to the questions of how many generators do we end up with; does this number depend on the vectors that we start with, or on the choices we make in removing superfluous generators.

Theorem 3.3. *Let* $S$ *be a subspace of a vector space* $V$ *over a field* $\mathbb{F}$. *Suppose* $S$ *is generated by* $n$ *vectors* $\mathbf{v}_1, \dots, \mathbf{v}_n$. *Let* $\{\mathbf{w}_1, \dots, \mathbf{w}_m\}$ *be a set of* $m$ *vectors in* $S$ *with* $m > n$. *Then the vectors* $\{\mathbf{w}_1, \dots, \mathbf{w}_m\}$ *are linearly dependent.*

*Proof.* We will do a proof by induction.

Suppose $n = 1$. Then each element of $S$ is of the form $\lambda \mathbf{v}_1$ for some $\lambda \in \mathbb{F}$. So there must be a choice of scalars $\lambda_1, \dots, \lambda_m$ so that

$$\mathbf{w}_i = \lambda_i \mathbf{v}_1 \quad , \quad i = 1, \dots, m$$

But then

$$-\lambda \mathbf{v}_1 + \mathbf{w}_1 = \mathbf{0}_V$$

is a dependence relation amongst the $\mathbf{w}_1, \dots, \mathbf{w}_m$ and so $\{w_1, \dots, w_m\}$ are linearly dependent.

Now assume the statement is true for $n = N - 1$. Write each $w_i$, $i = 1, \ldots, m > N$, as

$$\mathbf{w}_i = \sum_{j=1}^{N} \alpha_{ij} \mathbf{v}_j$$

Case 1: all $\alpha_{iN} = 0_{\mathbb{F}}$. In this situation each $\mathbf{w}_i$ lies in the span of the first $N - 1$ $\mathbf{v}_i$, and so the conclusion that $\{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$ are linearly dependent is affirmed by the induction hypothesis.

Case 2: At least one $\alpha_{iN} \neq 0_{\mathbb{F}}$. By, if necessary, reordering the $\mathbf{v}_i$, we can assume that $\alpha_{1N} \neq 0_{\mathbb{F}}$. Now consider

$$\mathbf{w}_2 - \frac{\alpha_{2N}}{\alpha_{1N}} \mathbf{w}_1$$

The coeffient of $v_N$ is this expression is, by construction, equal to $0_{\mathbb{F}}$. Similarly, the vectors

$$\begin{gathered} \mathbf{w}_3 - \frac{\alpha_{3N}}{\alpha_{1N}} \mathbf{w}_1 \\ \vdots \\ \mathbf{w}_m - \frac{\alpha_{mN}}{\alpha_{1N}} \mathbf{w}_1 \end{gathered}$$

all have $0_F$ component along $\mathbf{v}_N$. Thus, the $m - 1$ vectors $\mathbf{w}_2 - \frac{\alpha_{2N}}{\alpha_{1N}} \mathbf{w}_1, \ldots, \mathbf{w}_m - \frac{\alpha_{mN}}{\alpha_{1N}} \mathbf{w}_1$, all belong to the subspace generated by $\mathbf{v}_1, \ldots, \mathbf{v}_{N-1}$. Since $m > N$ implies $m - 1 > N - 1$, the induction hypothesis implies that these $m - 1$ vectors must be linearly dependents. So there are constants $\beta_2, \ldots, \beta_m$, not all equal to $0_{\mathbb{F}}$ such that

$$\beta_2 \left( \mathbf{w}_2 - \frac{\alpha_{2N}}{\alpha_{1N}} \mathbf{w}_1 \right) + \cdots + \beta_m \left( \mathbf{w}_m - \frac{\alpha_{mN}}{\alpha_{1N}} \mathbf{w}_m \right) = \mathbf{0}_V$$

or

$$-\left( \frac{\beta_2 \alpha_{2N}}{\alpha_{1N}} + \cdots + \frac{\beta_m \alpha_{mN}}{\alpha_{1N}} \right) \mathbf{w}_1 + \beta_2 \mathbf{w}_2 + \cdots + \beta_m \mathbf{w}_m = 0$$

which is a dependence relation amongst $\{w_1, \ldots, w_m\}$ since at least one of the coefficeints of the last $m - 1$ terms must be non-zero. □

COROLLARY 3.4. *Suppose* $\{v_1, \ldots, v_n\}$ *and* $\{w_1, \ldots, w_m\}$ *are two sets of generators of a subspace* $S$ *and that both sets of generators are linearly independent. Then* $n = m$.

*Proof.* If $m > n$, then we will would have more vectors in the set $\{w_1, \ldots, w_m\}$ than we have generators in set $\{v_1, \ldots, v_n\}$. By the preceding theorem, we would conclude that the set $\{w_1, \ldots, w_m\}$ must be a linearly dependent set of vectors. But that violates our hypothesis. If $n > m$, the a similar argument, in which the roles of the sets $\{v_1, \ldots, v_n\}$ and $\{w_1, \ldots, w_m\}$ are switched would force us to conclude that the vectors $\{v_1, \ldots, v_n\}$ are linearly independent, in violation of our hypothesis. The only other possibility left is $n = m$ and so the statement is proved. □

Thus the number of linearly independent vectors used to generate a subspace is independent of the choice of a linearly independent set of generators. This is an important *invariant* of a subspace and the motivation for the following definition.[1]

DEFINITION 3.5. *The common cardinality of any linearly independent set of generators for a subspace* $S$ *is called the* ***dimension*** *of* $S$.

DEFINITION 3.6. *A* ***basis*** *for a subspace* $S$ *is a linearly independent set of generators for* $S$.

So, the dimension of a subspace $S$ is the number of vectors in any basis of $S$.

[1]A notion that depends ostensibly on some choices (in a present case the choice of a linearly independent set of generators), but which in fact is independent of the choices made, we sometimes refer to as an *invariant* of the construction.