MATH 4063-5023
Homework Set 8

1. Determine if the following matrices $\mathbf{A}$ are diagonalizable. When $\mathbf{A}$ is diagonalizable, provide both the matrix $\mathbf{C}$ diagonalizing $\mathbf{A}$ and its diagonal form $\mathbf{D}$ (so that $\mathbf{D} = \mathbf{C}^{-1}\mathbf{A}\mathbf{C}$).

(a) $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix}$

(b) $\mathbf{A} = \begin{pmatrix} 1 & -2 \\ 1 & -1 \end{pmatrix}$

(c) $\mathbf{A} = \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}$

2. Consider the operator $T = \frac{d}{dx}$ acting on the vector space $\mathcal{P}_2$ of polynomials of degree ≤ 2.

(a) What are the eigenvalues and eigenvectors of T?

(b) Is the operator $\frac{d}{dx} : \mathcal{P}_2 \to \mathcal{P}_2$ diagonalizable?

(c) What is the minimal polynomial of T?

(d) What is the direct sum decomposition of $\mathcal{P}_2$ appropriate for T (as in Theorem 17.17). (Hint: look at Theorem 17.19 before trying to draw conclusions.)

3. Let V_1 and V_2 be non-zero subspaces of a ector space V. Show that $V = V_1 \oplus V_2$ if and only if $V = V_1 + V_2$ and $V_1 \cap V_2 = \{\mathbf{0}\}$.

4. Let $T \in L(V, V)$ be a linear transformation such that $T^2 = \mathbf{1}_{L(V,V)}$. Prove that $V = V_+ \oplus V_-$ where

$$V_\pm = \{v \in V \mid Tv = \pm v\} \quad .$$

5. Let V be a vector space over an algebraically closed field (e.g. $\mathbb{C}$) and let $T \in L(V, V)$ be a linear transformation whose only eigenvalue is 0. Prove that T is nilpotent; i.e $T^n = 0$ for some positive integer n.