MATH 4063-5023
Homework Set 4

1. Let $\mathcal{P}$ be the vector space of polynomials with indeterminant x.Which of the following mappings are linear transformations from $\mathcal{P}$ to itself

(a) $T : p \to xp$

(b) $T : p \to 2p$

(c) $T : p \to \frac{dp}{dx} + 2p$

(d) $T : p \to \int_0^1 p(x)\, dx$

2. Suppose $f : V \to W$ is a linear transformation:

(a) Prove that f is injective if and only if $\ker(f) = \{\mathbf{0}_V\}$.

(b) Prove that f is surjective if and only if $\dim(\mathrm{Im}(f)) = \dim W$.

(c) Prove that if f is bijective if and only if $\dim(V) = \dim(W)$ and $\ker(f) = \{\mathbf{0}_V\}$.

3. Prove that the composition $f \circ g$ of two linear transformations is a linear transformation.

4. Consider the mapping $T : \mathbb{R}^2 \to \mathbb{R}^3 \quad T([x_1, x_2]) = [x_1 - x_2, x_1 + x_2, x_1 - 2x_2]$

(a) Show that T is a linear transformation.

(b) Find the matrix corresponding to T and the natural bases of $B = \{[1,0], [0,1]\}$ and $B' = \{[1,0,0], [0,1,0], [0,0,1]\}$ of, respectively, $\mathbb{R}^2$ and $\mathbb{R}^3$.

(c) What is the kernel of this linear transformation.

(d) What is the range (i.e. the image) of this linear transformataion.

5. Let $\mathcal{P}_3$ be the vector space of polyonomials of degree ≤ 3 with natural basis $\{x^3, x^2, x, 1\}$. Find the matrix $T_{B,B}$ corresponding to the linear transformation

$$T : \mathcal{P}_3 \to \mathcal{P}_3 \quad , \quad p \to 2x\frac{d}{dx}p + p$$

and the basis B (same basis for the domain and codomain of T).

6. Suppose $f : V \to W$ is a linear transformation and S is a subspace of W contained in $\mathrm{Im}(f)$. Prove that

$$f^{-1}(S) \equiv \{v \in V \mid f(v) \in S\}$$

is a subspace of V.

7. Let S be a subspace of a vector space V over a field $\mathbb{F}$ and let V/S be the corresponding quotient space:

$$V/S := \{v + S \mid v \in V\}$$

where

$$v + S := \{v' \in V \mid v' = v + s \quad \text{for some } s \in S\}$$

Let addition and scalar multiplication of elements of V/S be defined by

$$\begin{array}{rcl} + & : & V/S \times V/S \to V/S \quad ; \quad (v + S) + (w + S) := (v + w + S) \\ * & : & \mathbb{F} \times V/S \to V/S \quad ; \quad \lambda(v + S) := (\lambda v + S) \end{array}$$

Show that V/S is a vectors space over $\mathbb{F}$ (i.e., verify all 8 axioms for a vector space).

8. Let S be the subspace of $\mathbb{R}^3$ spanned by $[1,0,0]$ and $[0,1,0]$. Identify let $\mathbf{v}_1 = [1,-1,3]$ and let $\mathbf{v}_2 = [2,3,1]$. Determine $[\mathbf{v}_1]_S + [\mathbf{v}_2]_S$ explicitly (it has to be some hyperplane in the direction of S inside $\mathbb{R}^3$).