MATH 4063-5023
Homework Set 1

1. Let $\mathbb{F}$ be a field, and let $\mathbb{F}^n$ denote the set of n-tuples of elements of $\mathbb{F}$, with operations of scalar multiplication and vector addition defined by

$$\begin{aligned}
\lambda \cdot [\alpha_1, \ldots, \alpha_n] \quad &: \; = [\lambda a_1, \ldots, \lambda a_n] \qquad , \qquad \text{for all } \lambda \in \mathbb{F} \text{ and all } [\alpha_1, \ldots, \alpha_n] \text{ in } \mathbb{F}^n \\
[\alpha_1, \ldots, \alpha_n] + [\beta_1, \ldots, \beta_n] \quad &: \; = [\alpha_1 + \beta_i, \ldots, \alpha_n + \beta_n] \qquad , \qquad \text{for all } [\alpha_1, \ldots, \alpha_n] \text{ and } [\beta_1, \ldots, \beta_n] \text{ in } \mathbb{F}^n
\end{aligned}$$

Check that $\mathbb{F}^n$ satisfies all the axioms of a vector space over $\mathbb{F}$.

2. Let $\mathcal{C}^1(\mathbb{R})$ be the set of continuous, differentiable functions on the real line with values in $\mathbb{R}$. Define scalar multiplication and vector addition on $\mathcal{C}(\mathbb{R})$ by

$$\begin{aligned}
(\lambda \cdot f)(x) \quad &: \; = \lambda f(x) \qquad , \qquad \forall \lambda \in \mathbb{R} \quad , \quad \forall f \in \mathcal{C}^1(\mathbb{R}) ; \\
(f + g)(x) \quad &: \; = f(x) + g(x) \qquad , \qquad \forall f, g \in \mathcal{C}^1(\mathbb{R}) \; .
\end{aligned}$$

Check that $\mathcal{C}^1(\mathbb{R})$ satisfies the axioms for a vector space over $\mathbb{R}$.

3. Determine which of the following subsets are subspaces of $\mathcal{C}^1(\mathbb{R})$

(a) The set of polynomial functions in $\mathbb{C}^1(\mathbb{R})$.
(b) The set of all functions $f \in \mathcal{C}^1(\mathbb{R})$ such that $f\left(\frac{1}{2}\right)$ is a rational number.
(c) The set of all $f \in \mathcal{C}^1(\mathbb{R})$ such that $f\left(\frac{1}{2}\right) = 0$.
(d) The set of all $f \in \mathcal{C}^1(\mathbb{R})$ such that $\int_0^1 f(x)\,dx = 1$
(e) The set of all $f \in \mathcal{C}^1(\mathbb{R})$ such that $\int_0^1 f(x)\,dx = 0$
(f) The set of all $f \in \mathcal{C}^1(\mathbb{R})$ such that $\frac{df}{dt} = 0$.

4. Prove that a subspace (a subset of a vector space closed under vector addition and scalar multiplication) is a vector space by verifying all 8 axioms.

5. Is the intersection of two subspaces a subspace (prove your answer)?

6. Is the union of two subspaces a subspace (explain your answer)?

7. Show that a set of vectors which contains a linearly dependent set of vectors is itself a linearly dependent set of vectors.

8. Let $\{v_1, \ldots, v_n\}$ be a basis for a (non-trivial) vector space V. Show that $v_i \neq \mathbf{0}_V$ for all $i = 1, \ldots, n$.

9. Let $\{v_1, \ldots, v_k\}$ be a linearly independent set of vectors. Let

$$\begin{aligned}
u &= \alpha_1 v_1 + \cdots + \alpha_k v_k \\
w &= \beta_1 v_1 + \cdots + \beta_k v_k
\end{aligned}$$

Prove that $u = w$ if and only if $\alpha_1 = \beta_1, \alpha_2 = \beta_2, \ldots, \alpha_k = \beta_k$.

10. Show that $\left\{1, x, x^2, \ldots, x^n\right\}$ is a basis for the vector space $\mathcal{P}_n$ of polynomials of degree $\leq n$. (Hint: just check that the definition of a basis is satisfied.)