LECTURE 12

The Completeness Axiom, Cont'd

AXIOM 12.1 (The Completeness Axiom). *Every non-empty subset S of $\mathbb{R}$ that is bounded from above has a least upper bound* $\sup(S) \in \mathbb{R}$.

THEOREM 12.1. *Every non-empty subset S of $\mathbb{R}$ that is bounded from below has a greatest lower bound $inf\, S$.*

Proof. Let T be the set $\{-s \mid s \in S\}$. Since S is bounded from below there is an $m \in \mathbb{R}$ such that $m \leq s$ for all $s \in S$. This implies $-s \leq -m$ for all $s \in S$ and so $t \leq -m$ for all $t \in T$. So T is bounded from above, hence by the Completeness Axiom, $sup\, T$ exists. Let $u = sup\, T$. We shall show that $-u = inf\, S$.

More precisely, we shall show that

$$-u \leq s \quad , \quad \forall\, s \in S \tag{12.1}$$

and that

$$t \leq s \quad , \quad \forall s \in S \quad \Rightarrow \quad t \leq -u \quad . \tag{12.2}$$

Now by definition, since u is the least upper bound of T,

$$-s \leq u \quad , \quad \forall\, s \in S \tag{12.3}$$

and

$$-s \leq q \quad , \quad \forall\, s \in S \quad \Rightarrow \quad u \leq q \tag{12.4}$$

Now from Theorem 3.2 (i) we know (12.3) is equivalent to (12.1). Setting $q = -t$, (12.4) reads

$$-s \leq -t \quad , \quad \forall\, s \in S \quad \Rightarrow \quad u \leq -t$$

or, using Theorem 3.4 (i) again,

$$t \leq s \quad , \quad \forall\, s \in S \quad \Rightarrow \quad t \leq -u$$

which is precisely (12.2). $\square$

THEOREM 12.2 (The Archimedian Property of $\mathbb{R}$). *The set $\mathbb{N}$ of natural numbers is unbounded from above in $\mathbb{R}$.*

Proof. (Proof by Contradiction). Suppose $\mathbb{N}$ is bounded from above in $\mathbb{R}$. Then by the Completeness Axiom, $\mathbb{N}$ has a least upper bound $m \in \mathbb{R}$. This implies that $m-1$ is not an upper bound for $\mathbb{N}$ (since there can be no upper bound smaller than m), hence the must be an element $n \in \mathbb{N}$ such that

$$m - 1 < n$$

But if $n \in \mathbb{N}$ then $n + 1 \in \mathbb{N}$ and so adding 1 to both sides of the above inequality yields

$$m = m - 1 + 1 < n + 1 \in \mathbb{N}$$

so m can not be an upper bound for $\mathbb{N}$ (let alone the least upper bound). We conclude that $m = \sup(\mathbb{N})$ does not exist.

THEOREM 12.3. *The following statements are equivalent to the Archimedian Property of $\mathbb{R}$.*

(1) *For each $z \in \mathbb{R}$, there exists $n \in \mathbb{N}$ such that $n > z$.*

(2) *If* $x > 0$ *and for each* $y \in \mathbb{R}$, *there exists an* $n \in \mathbb{N}$ *such that* $nx > y$.
(3) *For each* $x > 0$, *there exists* $n \in \mathbb{N}$ *such that* $0 < \frac{1}{n} < x$.

Proof.

(Archimedian Property $\Rightarrow$ 1). Suppose (1) is false. Then there exists a $z \in \mathbb{R}$, such that no $n \in \mathbb{N}$ is such that $n > z$; i.e., $n \leq z$ for all $n \in \mathbb{N}$, i.e. $\mathbb{N}$ has an upper bound in $\mathbb{R}$. Hence the Archimedian Property is false. Thus, the contrapositive of (Archimedian Property $\Rightarrow$ 1) is proven.

($1 \Rightarrow 2$). Let $z = y/x$. Then, if (1) is true, there exists $n \in \mathbb{N}$ such that $n > \frac{y}{x}$, or (using that fact that $x > 0$) that $nx > y$.

($2 \Rightarrow 3$). Suppose (2) is true. Then setting $y = 1$ we know there exists n such that $nx > 1$. Multiplying both sides of this last inequality by $1/n$ we have $\frac{1}{n} < x$. Also, $0 < \frac{1}{n}$ since if it were false, then $\frac{1}{n} \leq 0$. And this last ineqality when multiplied by the postive number n^2 would yield $n \leq 0$ which would mean that n was not a positive integer.

($3 \Rightarrow$ Archimedian Property). Suppose that $\mathbb{N}$ is bounded above by some real number m; i.e., $n < m$ for all $n \in \mathbb{N}$. Then

$$\frac{1}{m} < \frac{1}{n} \qquad \forall n \in \mathbb{N}$$

which contradicts (3) (because there'd be no $1/n$ between 0 and $\frac{1}{m}$. Thus the compositive of ($3 \Rightarrow$ Archimedian Property) is proven.

THEOREM 12.4. *(The Denseness of* $\mathbb{Q}$*) If* $a, b \in \mathbb{R}$ *and* $a < b$, *then there is a rational number* r *such that* $a < r < b$.

Proof. It suffices to show that there exist integers m and $n > 0$ such that

$$na < m < nb \quad .$$

Since $0 < b - a$, the Statement (2) of the preceding theorem tells us that there exists an $n \in \mathbb{N}$ such that

$$1 < n(b - a)$$

or

$$an + 1 < bn \quad .$$

At this point it *seems* obvious that there is an integer lying between an and bn. Rather than make a plausibility argument, we shall provide an explicit construction of such an integer.

By the Archimedean Property again, there also exists positive integers k', k'' such that

$$|an| < k' \qquad , \qquad |bn| < k'' \quad .$$

Set

$$k = max\{k', k''\} \quad .$$

Then

$$-k < an < bn < k \quad .$$

The set

$$\{j \in \mathbb{Z} \mid -k < j < k \text{ and } an < j\}$$

is finite and nonempty. Set

$$m = min\,\{j \in \mathbb{Z} \mid -k < j < k \text{ and } an < j\}$$

so that

$$an < m \quad , \quad \text{but } m - 1 \leq an \quad .$$

Then we have

$$m = (m-1) + 1 \leq an + 1 < bn \quad .$$

Now we have found an $m \in \mathbb{Z}$ such that

$$an < m < bn \quad .$$

□