LECTURE 11

Fields and Ordered Fields, Cont'd

DEFINITION 11.1. *Let F be an ordered field. If $a \in F$, then the* ***absolute value*** *$|a|$ of a is the element of F defined by*

$$|a| = \left\{ \begin{array}{ccc} a & , & if 0 \preceq a \\ -a & , & if a \preceq 0 \end{array} \right. .$$

DEFINITION 11.2. *Let a, b be arbitrary elements of an ordered field F. The* ***distance between*** *a* ***and*** *b is the element $dist(a,b) \in F$ defined by*

$$dist(a,b) = |a-b| \quad .$$

THEOREM 11.3. *Let F be an ordered field. Then:*

(i) $0 \preceq |a|$ *for all* $a \in F$.
(ii) $|ab| = |a| \cdot |b|$ *for all* $a, b \in F$.
(iii) $|a+b| \preceq |a| + |b|$ *for all* $a, b \in F$.

Proof.

(i) This obvious from the definition of $|a|$.
(ii) There are four easy cases here.

If $0 \preceq a$ and $0 \preceq b$, then $0 \preceq ab$ by Theorem 3.2 (**??**). So

$$|ab| = ab = |a|\,|b|$$

by Definition 3.3.

If $a \preceq 0$ and $b \preceq 0$, then $0 \preceq ab$ by Theorem 3.2 (ii). So

$$|ab| = ab = (-|a|)\,(-|b|) = |a|\,|b|$$

by Definition 3.3.

If $a \preceq 0$ and $0 \preceq b$, then $ab \preceq 0$ by Theorem 3.2 (ii). So

$$|ab| = -ab = -\left(-|a|\right)|b| = |a|\,|b|$$

by Definition 3.3.

If $0 \preceq a$ and $b \preceq 0$, then $ab \preceq 0$ by Theorem 3.2 (ii). So

$$|ab| = -ab = -|a|\left(|b|\right)| = |a|\,|b|$$

by Definition 3.3.

(iii) The inequalities

$$-|a| \preceq a \preceq |a|$$

follow from the facts that either $a = |a|$ or $a = -|a|$ and $-|a| \prec |a|$. Similarly,

$$-|b| \preceq b \preceq |b| \quad .$$

Applying Axiom O4 four times we get

$$-|a| + (-|b|) \preceq -|a| + b \preceq a + b \preceq a + |b| \preceq |a| + |b|$$

so that

$$-(|a|+|b|) \preceq a+b \tag{11.1}$$

and

$$a+b \leq |a|+|b| \tag{11.2}$$

Now (11.1) implies

$$-(a+b) \preceq |a|+|b| \quad . \tag{11.3}$$

Since $|a+b|$ is either equal to $a+b$ or $-(a+b)$, (11.2) and (3.3) imply

$$|a+b| \preceq |a|+|b| \quad .$$

∎

COROLLARY 11.4. *If F is an ordered field, then*

$$dist(a,c) \preceq dist(a,b) + dist(a,c) \quad ; \quad \forall\, a,b,c \in F \quad .$$

Homework:

1. Problem 2.1.
2. Prove Theorem 3.1.
3. Problem 3.1.

1. The Completeness Axiom

Thus far, we have been fairly careful up in constructing the number fields $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ from the Peano Axioms. A rigorous construction of the real numbers from the rational numbers takes quite a bit of work, however. Much more work than we have time for in this course. However, it will be important for us to understand the property of the real numbers that distinguishes the reals from the rationals; for it is precisely this property of the reals that permits the rigorous development of calculus. So, henceforth we shall not worry about the problem of *constructing* the set of real numbers, we shall simply assume that the real numbers exist and lay down as axioms the properties of $\mathbb{R}$ that we need for analysis.

DEFINITION 11.5. *Let S be a nonempty subset of $\mathbb{R}$.*

1. *"(a)" If s_o is an element of S with the property that*

$$s \in S \quad \Rightarrow \quad s \leq s_o$$

then s_o is called the ***maximum*** *of S.*

2. *"(b)" If s_o is an element of S with the property that*

$$s \in S \quad \Rightarrow \quad s_o \leq s$$

then s_o is called the ***minimum*** *of S.*

Examples:

1. "(a)" Every finite subset of $\mathbb{R}$ has a maximum. For example,

$$max\,\{1,3,6,-2\} = 6 \quad .$$

2. "(b)" Not every subset of $\mathbb{R}$ has a maximum. The subsets $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ have no maximum. The subset $\mathbb{N}$, however, does have a minimal element; viz., 1.

3. "(c)" The subset
$$\left\{x \in \mathbb{Q} \mid 1 \leq x \leq \sqrt{2}\right\}$$
has a minimum, 1, but it does not have a maximal element.
4. "(d)" The subset
$$\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\right\}$$
does not have a minimal element.

Below we present some definitions that will give us a handle on the counterexamples mentioned above.

DEFINITION 11.6. *Let S be a nonempty subset of $\mathbb{R}$.*

1. *"(a)" If a real number M has the property that*
$$s \leq M \quad , \quad \forall\, s \in S \quad ,$$
then M is called an ***upper bound*** *of S and the set S is said to be* ***bounded from above.***
2. *"(b)" If a real number m has the property that*
$$m \leq s \quad , \quad \forall\, s \in S \quad ,$$
then m is called a ***lower bound*** *of S and the set S is said to be* ***bounded from below.***
3. *"(c)" A subset $S \subset \mathbb{R}$ is said to be* ***bounded*** *if it is bounded from above and bounded from below. Thus, S is bounded if there exist real numbers m, M such that*
$$S \subset [m, M] \quad .$$

DEFINITION 11.7. *Let S be a nonempty subset of $\mathbb{R}$.*

1. *"(a)" Suppose S is bounded from above. The* ***suprenum****, or* ***least upper bound****, of S is the number $sup\ S \in \mathbb{R}$ defined by*
$$sup\ S = min\,\{x \in \mathbb{R} \mid x \text{ is an upper bound of } S\} \quad .$$
2. *"(b)" Suppose S is bounded form below. The* ***infimum*** *or* ***greatest lower upper bound****, of S is the number $inf\ S \in \mathbb{R}$ defined by*
$$sup\ S = max\,\{x \in \mathbb{R} \mid x \text{ is a lower bound of } S\} \quad .$$

There is a problem with this definition, however. We have already seen examples of subsets of $\mathbb{R}$ which do not have a maximum or a minimum; how, for example, do we know that the set
$$\{x \in \mathbb{R} \mid x \text{ is an upper bound of } S\}$$
has a minimal element?

This problem we shall resolve by regarding as an axiom the following of $\mathbb{R}$.

THEOREM 11.8. ***The Completeness Axiom*** *Every non-empty subset S of $\mathbb{R}$ that is bounded from above has a least upper bound.*